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Self-tuning average consensus in complex networks

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Abstract

In this paper we present an adaptive algorithm for distributed average consensus over a network of multi-agent systems. The coupling parameters defining the strength of agents interactions are locally self-tuned by each node based on the state information of its neighbors. Assuming that the underlying graph is connected, it is shown that the sequence of coupling parameters generated by normalized gradient algorithm (NGA) is convergent, and the agent states converge toward the average of the initial state values. Relation of the proposed method to synchronization phenomenon is discussed. Simulation results illustrate effectiveness of the proposed method.

1. Introduction

Emergence of a synchronized collective behavior in multi-agent systems is a topic of significant interest in various fields of science and engineering. Focal point of study in multi-agent coordination is understanding the consensus phenomenon where a number of autonomous agents reach a state of agreement, or synchronize certain state dependent quantity of interest, without central direction. An example of consensus phenomenon is the collective behavior in population of animals such as flocks, herds and swarms. The interaction or information exchange among agents is specified by a consensus algorithm, some times referred to as consensus protocol. Vicseck et al. [1] proposed a model where each agent is moving in the plane with constant speed and the heading equal to the average of headings

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of its neighbors. Simulation results presented in [1] demonstrate that the alignment is achieved, and asymptotically all agents move in the same direction. Mathematically rigorous explanation of the phenomenon observed in the above paper is given in [2,3]. One of the first study of consensus problem is presented in [4], where a model describing how a group might reach agreement on a common probability distribution is considered. Among the first considerations of consensus concepts in systems and control theory is distributed computation over networks [5], and asynchronous optimization algorithms [6]. For the last twenty years there have been a huge proliferation of interesting results covering variety of consensus related problems such as formation control, distributed optimization and task assignment, coordination in flocks and swarms, sensor fusion, distributed estimation and control. Many of those results consider issues of communication delay and presence of noise in exchanging information between agents, time-varying network topologies, asynchronous updating of agent states, and gossip algorithms. Since commenting on many important results in this area will take as much space as the main body of this paper, we confine ourself to pointing out survey papers and several monographs containing large amount of important references in this area. Large number of references are given in [7–9] as well as in the recent books [10–14]. A fairly comprehensive review of the results published since 2006 in control systems journals is given in [15], and its extended version [16] with over 300 references.

A separate track of large amount of contributions on emergence of spontaneous order in networked systems pursued not exclusively but mainly by the physicists' community is related to synchronizations phenomenon. Synchronization is an intriguing concept ubiquitous in biological, chemical, physical and social systems. Winfree [17] assumed that the coherent cooperative action can be modeled by a collection of interactive oscillatory agents. To model synchronization of biological clocks he proposed to use a population of interacting limit-cycle oscillators. Building on Winfree's work, Kuramoto [18] analytically studied conditions for spontaneous frequency synchronization of a group of phase coupled oscillators. Large number of references with impressive results on the subject of synchronization in oscillator networks can be found in recent surveys [19–21].

In this paper we consider the problem of adaptively tuning inter-agent coupling parameters so that the reached consensus value is equal to the average of initial conditions, hence the name average consensus. We analyze networked systems with identical dynamics represented by a discrete time integrator with unknown scalar parameter. Each agent locally tunes the coupling parameters so that the square of the error between agent state and the average of the states of its neighbors is minimized. Assuming that the underlying network graph is connected, it is proved that with the proposed self-tuning rule all agent states converge towards the same value equal to the average of the initial state values. We also show that in the special case of all-to-all coupling, average consensus can be achieved with a simple interacting function proportional to the error between respective agent state and the average of states of the rest of agents. In addition we discuss how the problem of frequency synchronization fits within the proposed self-tuning average consensus scheme.

The paper is organized as follows. Problem statement is given in Section 2. Global stability of self-tuning consensus is given in Section 3. Section 4 discusses relation of self-tuning consensus to Kuramoto synchronization. Simulation experiments are presented in Section 5.

Notation: The following notation is used throughout the paper:

- The abbreviation RHS means “right-hand side”.
- $\Re$ denotes the set of real numbers.
- The space of n-dimensional vectors with real elements is denoted by $\Re^n$.
- The superscript T denotes the transpose of a matrix.

- $\mathbf{I}$ represents the identity matrix.
- $\rho(\mathbf{A})$ denotes the spectral radius of a matrix $\mathbf{A}$.
- In this paper ℓ is used to denote a vector with all entries equal to one, i.e., $\ell^T=(1,1,\ldots,1)$.
- When performing majorizations and in certain upper bounds, we use c_i, $i=1,2,\ldots$ to denote nonnegative constants whose values are unimportant.
- $\|\mathbf{x}\|$ denotes Euclidean norm of a vector $\mathbf{x}$, and $\operatorname{sgn}(y)$ denotes the sign of a real number y.
- In this paper matrices are denoted by upper case boldface letters, and vectors are denoted by lower case boldface letters.

2. Problem formulation

Consider a group of N agents described by a discrete time dynamics given by

$$x_i(t+1)=x_i(t)+\beta(t)u_i(t), \quad x_i(0)=x_{i_0}, \quad i=1,\ldots,N \tag{1}$$

where time index $t\in\{0,1,2,\ldots\}$, $x_i(0)\in\Re$, and $u_i(t)\in\Re$ are the state and control signal of agent i at time t, respectively, $x_{i_0}\in\Re$, $1\le i\le N$ are initial states, while $\beta(t)$ is the unknown input gain. The communication topology of the above network of agents can be modeled by an undirected graph $\mathbf{G}=(\mathbf{V},\mathbf{E})$ with the set of nodes $\mathbf{V}=\{1,2,\ldots,N\}$ and the set of edges or communication links $\mathbf{E}\subseteq\mathbf{V}\times\mathbf{V}$. The node i represents the agent i, and the ordered pairs (i,j), $i\neq j$ denote edges where $(i,j)\in\mathbf{E}$, if and only if the i-th agent can directly receive information from the j-th agent. For undirected graph, if $(i,j)\in\mathbf{E}$ then $(j,i)\in\mathbf{E}$. The set of neighbors of node i is denoted by $\mathcal{N}_i=\{j\in\mathbf{V}|(i,j)\in\mathbf{E}\}$.

Model (1) can be viewed as a discrete time version of a group of identical continuous time agents whose dynamics is

$$\dot{x}_i(\tau)=\frac{1}{m}u_i(\tau), \quad x_i(0)=x_{i0},$$

where m is the mass, $u_i(\tau)$ is the external force driving the motion of the i-th agent, and $x_i(\tau)$ is its velocity. Such model can be of interest in analyzing flock behavior or considering the problem of achieving common velocity in a formation of unmanned aerial vehicles. In this case due to the fuel consumption during flight, mass m can be a slowly time-varying quantity. The parameter $\beta(t)$ in Eq. (1) can be interpreted as the inverse of mass m. Another situation where model (1) is of interest is the rendezvous problem with the objective for all vehicles to meet at a common location using only relative position information [10,11]. The dynamics of the i-th vehicle can be described by a simple discrete-time model given by Eq. (1). In this case the input gain $\beta(t)$ can be taken to be a constant, $\beta(t)=\beta$. Due to uncertainties in the control actuator dynamics, β can be considered an unknown parameter.

Each agent generates control sequence $\{u_i(t)\}$ so that all state variables asymptotically reach an agreement, i.e., $\lim_{t\to\infty}x_i(t)=x_c$, $i\in\mathbf{V}$, for some $x_c\in\Re$. We then say that the network of agents has reached consensus, and refer to x_c as the consensus value. Normally $u_i(t)$ is a function of the i-th agent state and the states of its neighbors. The network theory literature often refers to $u_i(t)$ as the *consensus protocol*. If $x_c=(1/N)\sum_{i=1}^{N}x_i(0)$, we say that the network achieves average consensus. In this paper we consider the following protocol:

$$u_i(t)=\sum_{j\in\mathcal{N}_i}\theta_{ij}(t)(x_j(t)-x_i(t)), \quad i\in\mathbf{V} \tag{2}$$

where $\theta_{ij}(t) \in \Re$ are interagent coupling parameters to be determined so that the agent states achieve average consensus. Note that the control signal given in Eq. (2) can be written in the form

$$u_i(t) = \boldsymbol{\theta}_i(t)^T \boldsymbol{\varphi}_i(t), \quad i \in \mathbf{V} \tag{3}$$

where

$$\boldsymbol{\theta}_i(t)^T = [\theta_{i1}\mathcal{I}_{i1}, \ldots, \theta_{iN}\mathcal{I}_{iN}] \tag{4}$$

and

$$\boldsymbol{\varphi}_i(t)^T = [\epsilon_{i1}, \ldots, \epsilon_{iN}] \tag{5}$$

where $\epsilon_{ij}(t) = \left(x_j(t) - x_i(t)\right)\mathcal{I}_{ij}$, and $\mathcal{I}_{ij}$ is an indicator function given by $\mathcal{I}_{ij} = 1$ if $j \in \mathcal{N}_i$, and $\mathcal{I}_{ij} = 0$ if $j \notin \mathcal{N}_i$. After substituting Eq. (3) in Eq. (1) we derive

$$x_i(t+1) = x_i(t) + \beta(t)\boldsymbol{\theta}_i(t)^T \boldsymbol{\varphi}_i(t), \quad i \in \mathbf{V}. \tag{6}$$

Define

$$\mathbf{x}(t)^T = [x_1(t), \ldots., x_N(t)]. \tag{7}$$

Then Eq. (6) can be written in the compact form

$$\mathbf{x}(t+1) = \mathbf{W}(t)\mathbf{x}(t) \tag{8}$$

where $\mathbf{W}(t)$ is an $N \times N$ matrix given by

$$\mathbf{W}(t) = \left[w_{ij}(t)\right], \quad w_{ij}(t) = \begin{cases} \beta(t)\theta_{ij}(t), & j \in \mathcal{N}_i \\ 1 - \beta(t) \sum\limits_{j \in \mathcal{N}_i} \theta_{ij}(t), & j = i \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

It is well-known that the condition for the iteration in Eq. (8) to converge to the average of initial states is $\ell^T = (1, \ldots, 1)$ to be left and right eigenvector of $\mathbf{W}(t)$ corresponding to the eigenvalue $\lambda_1 = 1$, i.e.,

$$\ell^T \mathbf{W}(t) = \ell^T \quad \text{and} \quad \mathbf{W}(t)\ell = \ell. \tag{10}$$

Then, the sum of the states is time invariant, i.e. $\ell^T \mathbf{x}(t+1) = \ell^T \mathbf{x}(t) = \cdots = \ell^T \mathbf{x}(0)$, and ℓ (or scalar multiple of it) is a fixed point of the recursion defined by Eq. (8).

In this paper we satisfy conditions given in Eq. (10) by generating symmetric matrix $\mathbf{W}(t)$, $t \geq 0$. Each agent tunes its parameter vector $\boldsymbol{\theta}_i(t)$ by minimizing the following cost function:

$$J_i(\boldsymbol{\theta}_i) = \tfrac{1}{2}(x_i(t+1) - \overline{x}_i(t+1))^2, \quad i \in \mathbf{V} \tag{11}$$

where $\overline{x}_i(t+1)$ represents the average of the i-th agent neighbors states including its own state,

$$\overline{x}_i(t+1) = \frac{1}{1+N_i} \sum_{j \in \mathcal{N}_i'} x_j(t), \quad \mathcal{N}_i' = \mathcal{N}_i \cup \{i\} \tag{12}$$

where N_i denotes cardinality of the set $\mathcal{N}_i$ (number of elements in $\mathcal{N}_i$).

Since from Eq. (6), $\partial x_i(t+1)/\partial \boldsymbol{\theta}_i(t) = \beta(t)\boldsymbol{\varphi}_i(t)$, gradient based minimization of the cost function given in Eq. (11) suggests the following updating rule for $\boldsymbol{\theta}_i(t)$:

$$\boldsymbol{\theta}_i(t+1) = \boldsymbol{\theta}_i(t) - \beta(t)\boldsymbol{\varphi}_i(t)e_i(t+1) \tag{13}$$

where

$$e_i(t+1) = x_i(t+1) - \overline{x}_i(t+1). \tag{14}$$

Since $\beta(t)$ is an unknown parameter, instead of Eq. (13), we use the following "normalized gradient" algorithm. Define

$$\boldsymbol{\theta}'_i(t)^T = [\theta'_{i1}\mathcal{I}_{i1}, \ldots, \theta'_{iN}\mathcal{I}_{iN}], \quad i = 1, \ldots, N.$$

Then the following scheme can be used to generate $\theta_i(t)$, $i \in V$,

$$\boldsymbol{\theta}'_i(t+1) = \boldsymbol{\theta}_i(t) - \frac{\mu_i}{r_i(t)}\mathrm{sgn}(\beta(t))\boldsymbol{\varphi}_i(t)e_i(t+1), \quad i \in \mathbf{V} \tag{15}$$

$$\theta_{ij}(t) = \tfrac{1}{2}\left(\theta'_{ij}(t)\mathcal{I}_{ij} + \theta'_{ji}(t)\mathcal{I}_{ji}\right), \quad j \in \mathcal{N}_i, \tag{16}$$

where $\theta_{ij}(t)$ are components of the vector $\boldsymbol{\theta}_i(t)$ defined by Eq. (4), $\mu_i > 0$ is the algorithm step size, and

$$r_i(t) = 1 + \|\boldsymbol{\varphi}_i(t)\|^2. \tag{17}$$

In order to assure that $\boldsymbol{\theta}_i(t)$ will be updated in the direction of the negative gradient of the cost function (11), in Eq. (15) it is assumed that the sign of $\beta(t)$ is known.

Note that the above algorithm generates the coupling parameter vector $\boldsymbol{\theta}_i(t)$ in two steps. First Eq. (15) gives $\boldsymbol{\theta}'_i(t)$, $i \in V$. Then according to Eq. (16) the components $\theta_{ij}(t)$ of $\boldsymbol{\theta}_i(t)$ are obtained by averaging respective elements of $\boldsymbol{\theta}'_i(t)$ and $\boldsymbol{\theta}'_j(t)$, $\forall j \in \mathcal{N}_i$, $i = 1, \ldots, N$. Observe that Eq. (16) guarantees that $\theta_{ji}(t) = \theta_{ij}(t)$, for all $i \neq j$. In other words $\mathbf{W}(t)$ in Eq. (8) is a symmetric matrix. Eq. (16) assumes that besides respective states $x_i(t)$ and $x_j(t)$, agents i and j exchange parameter values $\theta'_{ij}(t)$ and $\theta'_{ji}(t)$ as well. Iteration (15) starts with arbitrary initial conditions $\theta_{ij}(0)$ and $\theta_{ji}(0)$. The role of gradient normalizer $r_i(t)$ will become clear later when we consider global stability of the algorithm given by Eqs. (15)–(17).

From Eq. (15) it can be seen that $\theta_{ij}(t)$ is a function of $\{x_i(k+1)\}$, $1 \le k \le t$, and $\{x_j(k)\}$, $1 \le k \le t$, $j \in \mathcal{N}_i$. Also from Eqs. (1) and (2) it follows that $x_i(t+1)$ is a function of $\theta_{ij}(k), 0 \le k \le t$. Thus there is a highly nonlinear interplay between sequences $\{\theta_{ij}(t)\}$ and $\{x_i(t)\}$ implying that the interaction among agents is not a linear function.

We now comment on distributed nature of the proposed algorithm. From Eq. (2) it is obvious that each agent generates its control signal $u_i(t)$, $i \in V$ by using only states of its neighbors, i.e., states $x_j(t)$, $j \in \mathcal{N}_i$. Also from Eqs. (15) and (16) it is clear that the i-th agent generates coupling parameters $\theta_{ij}(t)$ (used in Eq. (2)) by using only variables obtained from its neighbors $j \in \mathcal{N}_i$. Therefore each agent is running its own local algorithm based on information available only from respective neighbors, without central coordination.

3. Global stability and convergence of the self-tuning consensus

We now examine stability and convergence of the multi-agent system defined by Eq. (6), and the accompanying algorithm given by Eqs. (15)–(17). For the sake of clarity, and in order to avoid tedious and cumbersome algebra, we will analyze the case of constant $\beta(t)$, i.e., in Eq. (1) we set $\beta(t) = \beta$ for all $t \ge 0$. The case of time-varying $\beta(t)$ will be analyzed in the follow up paper. In this section we show that the algorithm defined by Eqs. (15)–(17) along with the system dynamics given by Eq. (6) is globally stable in the sense that sequences $\{e_i(t)\}$ and $\{\boldsymbol{\varphi}_i(t)\}$, $t \ge 0$, $i \in \mathbf{V}$, have finite total energies, the sequence $\{\theta_{ik}(t)\}$, $t \ge 0$, $i, k \in \mathbf{V}$, $i \neq k$, is convergent as

$t \to \infty$, and all states $x_i(t)$ converge towards the same consensus value equal to the average of the initial states $x_i(0)$, $i = 1, \ldots, N$. By global we mean that all of the above claims hold for all initial conditions $x_i(0)$, and $\theta_{ij}(0)$, $i, j \in \mathbf{V}$. The above propositions are proved by assuming the following.

Assumption 1. The underlying network graph is connected.

Assumption 2. The sign of parameter β in Eq. (1), and the upper bound β_{max} of $|\beta|$ are known for each agent. The step size μ_i in Eq. (15) satisfies $\mu_i < 2/\beta_{max}$, for all $i \in \mathbf{V}$.

Define

$$\mathbf{e}(t)^T = [e_1(t), \ldots, e_N(t)], \tag{18}$$

and

$$\boldsymbol{\phi}_i(t) = \mathbf{x}(t) - x_i(t)\ell, \quad i \in \mathbf{V}. \tag{19}$$

The following lemma is useful for future reference.

Lemma 1. *Let Assumption 1 hold. Then for all $i = 1, \ldots, N$,*

$$\sum_{t=0}^{n} \| \boldsymbol{\phi}_i(t+1) \|^2 \le c_1 + c_2 \sum_{t=0}^{n} \| \mathbf{e}(t+1) \|^2, \quad \forall n \ge 0 \tag{20}$$

where $\mathbf{e}(t)$ is defined by Eq. (18), $\boldsymbol{\phi}_i(t)$ is defined by Eq. (19), while c_1 and c_2 are positive constants.

Proof. Note that $\bar{x}_i(t+1)$ given by Eq. (12) can be written in the form

$$\bar{x}_i(t+1) = \mathbf{a}_i^T \mathbf{x}(t) \tag{21}$$

where $\mathbf{x}(t)$ is the state vector defined by Eq. (7), and

$$\mathbf{a}_i^T = [a_{i1}, \ldots, a_{iN}], \quad a_{ij} = \begin{cases} \dfrac{1}{1+N_i}, & j = i \text{ and } j \in \mathcal{N}_i, \\ 0 & \text{otherwise.} \end{cases} \tag{22}$$

Then from Eqs. (14), (21), and (22), we have

$$\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + \mathbf{e}(t+1), \tag{23}$$

where $\mathbf{A}$ is an $N \times N$ matrix given by $\mathbf{A} = [a_{ij}]$, with elements a_{ij} defined by Eq. (22). Since

$$\mathbf{A}\ell = \ell, \quad \ell^T = (1, 1, \ldots, 1) \tag{24}$$

it follows that $\lambda_1 = 1$ is an eigenvalue of $\mathbf{A}$ with the corresponding right eigenvector equal to ℓ. Since $\mathbf{A}$ is a stochastic matrix, $\lambda_1 = 1$ is its maximal eigenvalue [22, p. 83]. By virtue of the fact that $\mathbf{G}$ is a connected graph, the nonnegative matrix $\mathbf{A}$ is irreducible (see for example [23, theorem 6.2.24, p. 362]) implying that $\lambda_1 = 1$ is an algebraically simple eigenvalue of $\mathbf{A}$ (see [23, theorem 8.4.4, p. 508]). Thus $\mathbf{A}$ is a primitive matrix. Consequently, except $\lambda_1 = 1$, the rest of the eigenvalues λ_i of the matrix $\mathbf{A}$ satisfy the condition $|\lambda_i| < 1$, $i = 2, \ldots, N$. Let $\mathbf{y}$ be the left eigenvector of $\mathbf{A}$ associated to λ_1, and normalized so that $\mathbf{y}^T \ell = 1$.

Based on this discussion we can decompose the matrix $\mathbf{A}$ as follows:

$$\mathbf{A} = \mathbf{A}_1 + \ell \mathbf{y}^T, \quad \ell^T \mathbf{y} = 1 \tag{25}$$

where all eigenvalues of $\mathbf{A}_1$ are inside the unit circle, i.e., the spectral radius of $\mathbf{A}_1$ satisfies $\rho(\mathbf{A}_1) < 1$. Hence from Eq. (23), one can obtain

$$\mathbf{x}(t+1) = \mathbf{A}_1\mathbf{x}(t) + \ell\mathbf{y}^T\mathbf{x}(t) + \mathbf{e}(t+1) \tag{26}$$

or

$$\mathbf{x}(t+1) = \mathbf{H}(q^{-1})\left[\ell\mathbf{y}^T\mathbf{x}(t) + \mathbf{e}(t+1)\right] \tag{27}$$

where

$$\mathbf{H}(q^{-1}) = (\mathbf{I} - q^{-1}\mathbf{A}_1)^{-1} \tag{28}$$

and q^{-1} is the unit delay operator, i.e., $q^{-1}\mathbf{x}(t+1) = \mathbf{x}(t)$. Since from Eqs. (24) and (25), $\mathbf{A}_1\ell = 0$, we have $\mathbf{H}(q^{-1})\ell = (\mathbf{I} + \sum_{k=1}^{\infty} q^{-k}\mathbf{A}_1^k)\ell = \ell$. Then Eq. (27) gives

$$\mathbf{x}(t+1) = \ell\mathbf{y}^T\mathbf{x}(t) + \mathbf{H}(q^{-1})\mathbf{e}(t+1). \tag{29}$$

From Eqs. (29) and (19) we can derive

$$\boldsymbol{\phi}_i(t+1) = \ell\mathbf{y}^T\mathbf{x}(t) + \mathbf{H}(q^{-1})\mathbf{e}(t+1) - \ell x_i(t+1) \tag{30}$$

or

$$\boldsymbol{\phi}_i(t+1) = \ell\mathbf{y}^T(\mathbf{x}(t) - \ell x_i(t)) + \ell\mathbf{y}^T\ell x_i(t) - \ell x_i(t+1) + \mathbf{H}(q^{-1})\mathbf{e}(t+1). \tag{31}$$

Since $\mathbf{y}^T\ell = 1$, we have

$$\boldsymbol{\phi}_i(t+1) = \ell\mathbf{y}^T\boldsymbol{\phi}_i(t) + \ell[x_i(t) - x_i(t+1)] + \mathbf{H}(q^{-1})\mathbf{e}(t+1). \tag{32}$$

Observe that Eq. (12) can be written as

$$\bar{x}_i(t+1) = x_i(t) + \mathbf{b}_i^T\boldsymbol{\phi}_i(t) \tag{33}$$

where $\mathbf{b}_i$ is defined as follows:

$$\mathbf{b}_i^T = [b_{i1}, \ldots, b_{iN}], \quad b_{ij} = \begin{cases} \dfrac{1}{1+N_i}, & j \in \mathcal{N}_i \\ 0 & \text{otherwise.} \end{cases} \tag{34}$$

Then the error $e_i(t+1)$ given by Eq. (14) can be expressed as

$$e_i(t+1) = x_i(t+1) - x_i(t) - \mathbf{b}_i^T\boldsymbol{\phi}_i(t). \tag{35}$$

Define the following matrix:

$$\mathbf{Q}_i = \ell(\mathbf{y} - \mathbf{b}_i)^T. \tag{36}$$

Then by using Eq. (35) in the second term on the RHS of Eq. (32) one can obtain

$$\boldsymbol{\phi}_i(t+1) = \mathbf{Q}_i\boldsymbol{\phi}_i(t) - \ell e_i(t+1) + \mathbf{H}(q^{-1})\mathbf{e}(t+1), \tag{37}$$

from where it follows that

$$\boldsymbol{\phi}_i(t+1) = \mathbf{Q}_i^t\boldsymbol{\phi}_i(0) + \sum_{k=0}^{t} \mathbf{Q}_i^{t-k}[\mathbf{H}(q^{-1})\mathbf{e}(k+1) - \ell e_i(k+1)]. \tag{38}$$

Note that the matrix $\mathbf{Q}_i$ in Eq. (36) is of rank 1 and its only nonzero eigenvalue is

$$\rho_1 = \ell^T(\mathbf{y} - \mathbf{b}_i) = 1 - \ell^T\mathbf{b}_i = 1 - \sum_{k=1}^{N} b_{ik} = 1 - \frac{N_i}{1+N_i} < 1.$$

Hence, there exists a positive constant c_3 such that $\|\mathbf{Q}_i^k\| \le c_3\rho_1^k$, $0<\rho_1<1$, for all $k\ge 0$ [24, p. 174]. Then from Eq. (38) it follows that for some positive constants c_4 and c_5,

$$\|\boldsymbol{\phi}_i(t+1)\|^2 \le c_4\rho_1^k + c_5\sum_{k=0}^{t}\rho_1^{t-k}\|\mathbf{e}(k+1)\|^2, \tag{39}$$

where we used the fact that $\mathbf{H}(q^{-1})$ is a stable operator, and $e_i(t)$ is absorbed by $\mathbf{e}(t)$. We say that the operator $\mathbf{H}(q^{-1})$ given by Eq. (28) is stable if the corresponding transfer function $\mathbf{H}(z^{-1})=(I-z^{-1}\mathbf{A}_1)^{-1}$ is stable, where z is a complex variable. The transfer function $\mathbf{H}(z^{-1})$ is stable by virtue of the fact that $\rho(\mathbf{A}_1)<1$, i.e., all eigenvalues of $\mathbf{A}_1$ are strictly inside the unit circle.

Summing up both sides of Eq. (39) from $t=0$ to $t=n$ yields Eq. (20). □

Next we can formulate global stability results.

Theorem 1. *Let Assumptions A1 and A2 hold. Then there exist some positive constants c_6 and c_7 so that for all finite initial conditions $x_i(0)$ and $\theta_{ij}(0)$, $1\le i,j\le N$, the multi-agent system given by Eq. (6) and the parameter estimation algorithm defined by Eqs. (15)–(17) provide*

$$(1)\quad \sum_{t=0}^{n}\|\mathbf{e}(t+1)\|^2 \le c_6<\infty, \quad n\ge 0, \tag{40}$$

$$(2)\quad \sum_{t=0}^{n}\|\boldsymbol{\phi}_i(t)\|^2 \le c_7<\infty, \quad \forall i\in\mathbf{V} \tag{41}$$

for all $n\ge 0$, where the vector $\boldsymbol{\phi}_i(t)$ is defined by Eq. (19).

(3) $\lim_{t\to\infty}\boldsymbol{\theta}_i(t)$ *exists,* $\forall i\in\mathbf{V}$

(4) *All agents converge toward the average of initial state values $x_i(0)$, i.e.,*

$$\lim_{t\to\infty}x_i(t)=\frac{\ell^T\mathbf{x}(0)}{N}, \quad i\in\mathbf{V} \tag{42}$$

where ℓ is the vector with all elements equal to one.

Proof. We first express the error $e_i(t+1)=x_i(t+1)-\overline{x}_i(t+1)$ in terms of the following parameter error:

$$\tilde{\boldsymbol{\theta}}_i(t)=\beta\boldsymbol{\theta}_i(t)-\mathbf{b}_i \tag{43}$$

where the vectors $\mathbf{b}_i$ are defined in Eq. (34). Note that Eq. (12) can be written in the form

$$\overline{x}_i(t+1)=x_i(t)+\mathbf{b}_i^T\boldsymbol{\varphi}_i(t) \tag{44}$$

where $\boldsymbol{\varphi}_i(t)$ is defined by Eq. (5). Then it is not difficult to see that from Eqs. (6), (14), (43), and (44), we can obtain

$$e_i(t+1)=\tilde{\boldsymbol{\theta}}_i^T\boldsymbol{\varphi}_i(t). \tag{45}$$

The parameter errors $\tilde{\boldsymbol{\theta}}_i(t)$ are in effect governed by the recursion given by Eq. (15). In light of definitions given by Eqs. (4), (5), and (43), from Eq. (15), we can write the following recursion for each component of $\tilde{\boldsymbol{\theta}}_i(t)$:

$$\tilde{\theta}'_{ij}(t+1)\mathcal{I}_{ij} = \tilde{\theta}_{ij}(t)\mathcal{I}_{ij} - \frac{\mu_i}{r_i(t)}|\beta|\epsilon_{ij}(t)e_i(t+1) \tag{46}$$

where $\tilde{\theta}'_{ij} = \beta\theta'_{ij}(t+1) - b_{ij},\ j \in \mathcal{N}_i$. After squaring both sides of Eq. (46) it follows that

$$\tilde{\theta}'_{ij}(t+1)^2\mathcal{I}_{ij} = \tilde{\theta}^2_{ij}\mathcal{I}_{ij} - \frac{2\mu_i}{r_i(t)}|\beta|\tilde{\theta}_{ij}(t)\epsilon_{ij}(t)\mathcal{I}_{ij}e_i(t+1) + \frac{\mu_i^2\beta^2}{r_i(t)^2}\epsilon_{ij}(t)^2 e_{ij}(t)^2, \tag{47}$$

for all $i \in \mathbf{V}$, $j \in \mathcal{N}_i$. Since Eq. (16) implies

$$\tilde{\theta}_{ij}(t+1)^2 \le \tfrac{1}{2}\left(\tilde{\theta}'_{ij}(t+1)^2 + \tilde{\theta}'_{ji}(t+1)^2\right),$$

from Eq. (47) one can derive

$$\begin{aligned}\tilde{\theta}_{ij}(t+1)^2\mathcal{I}_{ij} &\le \frac{1}{2}\left[\tilde{\theta}_{ij}(t)^2\mathcal{I}_{ij} + \tilde{\theta}_{ji}(t)^2\mathcal{I}_{ji}\right] \\ &\quad - \frac{\mu_i|\beta|}{r_i(t)}\tilde{\theta}_{ij}(t)\epsilon_{ij}(t)\mathcal{I}_{ij}e_i(t+1) - \frac{\mu_j|\beta|}{r_j(t)}\tilde{\theta}_{ji}(t)\epsilon_{ji}(t)\mathcal{I}_{ji}e_j(t+1) \\ &\quad + \frac{\mu_i^2\beta^2}{2r_i(t)^2}\epsilon_{ij}(t)^2 e_i(t+1)^2 + \frac{\mu_j^2\beta^2}{2r_j(t)^2}\epsilon_{ji}(t)^2 e_j(t+1)^2.\end{aligned} \tag{48}$$

Define

$$v(t) = \sum_{i=1}^{N}\|\tilde{\boldsymbol{\theta}}_i(t)\|^2. \tag{49}$$

Then by virtue of the fact that $\tilde{\theta}_{ij}(t) = \tilde{\theta}_{ji}(t)$, and by Eq. (4), $\|\tilde{\boldsymbol{\theta}}_i(t)\|^2 = \sum_{i=1}^{N}\tilde{\theta}_{ij}(t)^2\mathcal{I}_{ij}$, from Eq. (48) we can obtain

$$\begin{aligned}v(t+1) &\le v(t) - |\beta|\sum_{i=1}^{N}\frac{\mu_i}{r_i(t)}e_i(t+1)\sum_{j=1}^{N}\tilde{\theta}_{ij}(t)\epsilon_{ij}(t)\mathcal{I}_{ij} \\ &\quad - |\beta|\sum_{j=1}^{N}\frac{\mu_j}{r_j(t)}e_j(t+1)\sum_{i=1}^{N}\tilde{\theta}_{ji}(t)\epsilon_{ji}(t)\mathcal{I}_{ji} \\ &\quad + \frac{\beta^2}{2}\sum_{i=1}^{N}\frac{\mu_i^2}{r_i(t)^2}e_i(t+1)^2\sum_{j=1}^{N}\epsilon_{ij}(t)^2 \\ &\quad + \frac{\beta^2}{2}\sum_{j=1}^{N}\frac{\mu_j^2}{r_j(t)^2}e_j(t+1)^2\sum_{i=1}^{N}\epsilon_{ji}(t)^2.\end{aligned} \tag{50}$$

Observe that Eqs. (4) and (5) imply $\sum_{j=1}^{N}\tilde{\theta}_{ij}(t)\epsilon_{ij}\mathcal{I}_{ij} = \tilde{\boldsymbol{\theta}}_i(t)^T\boldsymbol{\varphi}_i(t)$ which together with Eq. (45) yields

$$\sum_{j=1}^{N}\tilde{\theta}_{ij}(t)\epsilon_{ij}(t)\mathcal{I}_{ij} = e_i(t+1), \quad i = 1,\ldots,N. \tag{51}$$

By using Eq. (51) in Eq. (50) it follows that

$$v(t+1) \le v(t) - 2|\beta|\sum_{i=1}^{N}\frac{\mu_i}{r_i(t)}e_i(t+1)^2 + \sum_{i=1}^{N}\frac{(\mu_i\beta)^2}{r_i(t)}\frac{\|\boldsymbol{\varphi}_i(t)\|^2}{r_i(t)}e_i(t+1)^2 \tag{52}$$

where we used the fact that from Eq. (5), $\sum_{j=1}^{N}\epsilon_{ij}(t)^2=\|\boldsymbol{\varphi}_i(t)\|^2$. Since by definition of $r_i(t)$ (see Eq. (17)), $\left(\|\boldsymbol{\varphi}_i(t)\|^2/r_i(t)\right)\le 1$, the third term on the RHS of Eq. (52) can be overbounded by $\sum_{i=1}^{N}(\mu_i\beta)^2 e_i(t+1)^2/r_i(t)$. Hence, from relation (52) we obtain

$$v(t+1)\le v(t)-2\sum_{i=1}^{N}\mu_i|\beta|\left(1-\frac{\mu_i|\beta|}{2}\right)\frac{e_i(t+1)^2}{r_i(t)}. \tag{53}$$

Summing up both sides of Eq. (53) from $t=0$ to $t=n$ gives

$$v(n+1)\le v(0)-2\sum_{t=0}^{n}\sum_{i=1}^{N}\mu_i|\beta|\left(1-\frac{\mu_i|\beta|}{2}\right)\frac{e_i(t+1)^2}{r_i(t)}. \tag{54}$$

Since by Assumption 2, the step size μ_i satisfies $1-(\mu_i|\beta|/2)\ge 0$, inequality (54) implies

$$\sum_{i=1}^{N}\sum_{t=0}^{n}\frac{e_i(t+1)^2}{r_i(t)}\le K_1<\infty, \tag{55}$$

for some positive constant K_1 dependent on $v(0)$, $|\beta|$ and μ_i, $i=1,\dots,N$. Define

$$\overline{r}(t)=1+\sum_{i=1}^{N}\sum_{k=1}^{t}\|\boldsymbol{\varphi}_i(k)\|^2, \tag{56}$$

where $\boldsymbol{\varphi}_i(k)$ is defined by Eq. (5). Since $\|\mathbf{e}(t+1)\|^2=\sum_{i=1}^{N}e_i(t+1)^2$, and using the fact that by Eqs. (17) and (56), $\overline{r}(t)\ge r_i(t)$ for $i=1,\dots,N$, relation (55) implies

$$\sum_{t=0}^{n}\frac{||\mathbf{e}(t+1)||^2}{\overline{r}(t)}\le K_1<\infty. \tag{57}$$

Next we analyze $\overline{r}(n)$, the denominator in Eq. (57). Since from Eqs. (5) and (19), $\|\boldsymbol{\varphi}_i(t)\|\le\|\boldsymbol{\phi}_i(t)\|,\ \forall t\ge 0$, Eq. (17) gives $r_i(t)\le 1+\|\boldsymbol{\phi}_i(t)\|^2$. Then relations (20) and (56) imply that for some positive constants c_8 and c_9,

$$\overline{r}(n)\le 1+\sum_{t=0}^{n}\sum_{i=1}^{N}\|\boldsymbol{\phi}_i(t)\|^2\le c_8+c_9\sum_{t=0}^{n}\|\mathbf{e}(t+1)\|^2, \tag{58}$$

which together with Eq. (57) gives

$$\sum_{t=0}^{n}\frac{\|\mathbf{e}(t+1)\|^2}{c_8+c_9\sum_{m=0}^{t}\|\mathbf{e}(m+1)\|^2}\le K_1<\infty, \tag{59}$$

for all $n\ge 0$. We now demonstrate by contradiction that $\sum_{m=0}^{t}\|e(m+1)\|^2$ is bounded for all $t\ge 0$. Assume that $\sum_{m=0}^{t}\|e(m+1)\|^2\to\infty$ as $t\to\infty$. Then by Kronecker's Lemma (for convenience it is given in the Appendix) from Eq. (59), one can derive

$$\lim_{n\to\infty}\left(\sum_{t=0}^{n}\mathbf{e}(t+1)^2\right)\Bigg/\left(c_8+c_9\sum_{t=0}^{n}\|\mathbf{e}(t+1)\|^2\right)=0, \tag{60}$$

for some positive constants c_8 and c_9. Statement 1 of this theorem directly follows from the last equation. Relation (41) is a consequence of Eq. (40) and Lemma 1. We now prove that the parameter sequence $\{\theta_{ij}(t)\}$, $t\ge 0$, $1\le i,j\le N$, has a limit. From Eqs. (16) and (46), we can write

$$\tilde{\theta}_{ij}(t+1)\mathcal{I}_{ij}=\frac{1}{2}\left[\tilde{\theta}'_{ij}(t+1)+\tilde{\theta}'_{ji}(t+1)\right]\mathcal{I}_{ij}=\tilde{\theta}_{ij}(t)\mathcal{I}_{ij}$$

$$-\frac{1}{2}|\beta|\left[\frac{\mu_i}{r_i(t)}\epsilon_{ij}(t)e_i(t+1)+\frac{\mu_j}{r_j(t)}\epsilon_{ji}(t)e_j(t+1)\right]. \tag{61}$$

After summing both sides of Eq. (61) from $t=0$ to $t=n$, it follows that

$$\tilde{\theta}_{ij}(t+1)\mathcal{I}_{ij}=\tilde{\theta}_{ij}(0)\mathcal{I}_{ij}-\frac{1}{2}|\beta|\sum_{t=0}^{n}\left[\frac{\mu_i}{r_i(t)}\epsilon_{ij}(t)e_i(t+1)+\frac{\mu_j}{r_j(t)}\epsilon_{ji}(t)e_j(t+1)\right]. \tag{62}$$

Consider now an infinite series R_{ij} defined by

$$R_{ij}=\sum_{t=0}^{\infty}\frac{\epsilon_{ij}(t)e_i(t+1)}{r_i(t)}. \tag{63}$$

Since by Eqs. (5) and (19), $|\epsilon_{ij}(t)|\le\|\boldsymbol{\varphi}_i(t)\|\le\|\boldsymbol{\phi}_i(t)\|$, we have

$$\sum_{t=0}^{\infty}\left|\frac{\epsilon_{ij}(t)e_i(t+1)}{r_i(t)}\right|\le\sum_{t=0}^{\infty}\|\boldsymbol{\phi}_i(t)\|\,|e_i(t+1)|, \tag{64}$$

where we used the fact that by Eq. (17), $r_i(t)\ge 1$. Then from Eqs. (63), (64), and Cauchy–Schwartz's inequality, it follows that

$$|R_{ij}|\le\left(\sum_{t=0}^{\infty}\|\boldsymbol{\phi}_i(t)\|^2\right)^{1/2}\left(\sum_{t=0}^{n}e_i(t+1)^2\right)^{1/2}\le c_{10}<\infty \tag{65}$$

for $1\le i,j\le N$. Thus the infinite series R_{ij} is absolutely convergent. Hence, Eq. (62) implies that $\lim_{n\to\infty}\tilde{\theta}_{ij}(n)$ exists. It is left to prove the statement in Eq. (42) of the theorem. Since by construction (see Eq. (16)) matrix $\mathbf{W}(t)$ given by Eq. (9) is symmetric, it is not difficult to see that $\mathbf{W}(t)\ell=\ell$, and $\ell^T\mathbf{W}(t)=\ell^T$, $\forall t\ge 0$, where $\ell^T=(1,\dots,1)$. Then from Eq. (8) it follows that the sum of initial states $x_i(0)$, $i=1,\dots,N$ is time invariant, i.e.,

$$\ell^T\mathbf{x}(t+1)=\ell^T\mathbf{x}(t)=\cdots=\ell^T\mathbf{x}(0). \tag{66}$$

On the other hand the definition in Eq. (19) and the statement (41) give $\lim_{t\to\infty}(\mathbf{x}(t)-x_i(t)\ell)=0$, or

$$\lim_{t\to\infty}\left(\ell^T\mathbf{x}(t)-x_i(t)\ell^T\ell\right)=0,\quad i\in\mathbf{V}. \tag{67}$$

Since $\ell^T\ell=N$, Eqs. (66) and (67) imply that $\lim_{t\to\infty}x_i(t)=(1/N)\ell^T\mathbf{x}(0)$, $i=1,\dots,N$. Thus the theorem is proved. □

Let us comment on the role of normalizer $r_i(t)$ in Eq. (15). Assume for a moment that instead of Eq. (17), $r_i(t)$ is equal to one. Then if at some time instant $\|\boldsymbol{\varphi}_i(t)\|^2$ becomes too large, the third term on the RHS of Eq. (52) can dominate the second term, and $v(t+1)$ may not be non-increasing function of time t. In combination with appropriate choice of the step size μ_i, the normalizer $r_i(t)$ guarantees that the second term on the RHS of Eq. (52) is larger than the third term, and thus $v(t+1)\le v(t)$ for all $t\ge 0$ (see relation (53)).

Remark 1. In Eq. (15) we assumed that the sign of parameter β is known. This parameter represents the input gain, often referred to as the high-frequency gain of agent dynamics. Similarly as in the case of adaptive control systems, it is not uncommon to assume that the sign of β is a priory knowledge available to the designer (see for example [25, p. 193], or [26,

p. 332]). In case of a previously mentioned kinematic model $\dot{x}_i(\tau) = (1/m)u_i(\tau)$, parameter β is equal to $(1/m)$ where m is the mass. Obviously we can assume that $\mathrm{sgn}(1/m) > 0$.

Remark 2. For simplicity of presentation, in the previous proof we assumed that parameter β is a constant. The case of time-varying β can be analyzed by using techniques similar to [27] which is different than the analysis presented in this paper. In general, statements (40) and (41) cannot be derived in the case of time-varying β. For example if we assume that $|\beta(t) - \beta(t-1)| \le \alpha$, and $|\beta(t)| \le \beta_{max} < \infty$, $\forall t \ge 1$, then instead of Eqs. (40) and (41) it would be possible to demonstrate that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{t=0}^{n} \| \mathbf{e}(t+1) \|^2 \le c_{11}\alpha^2$$

and

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{t=0}^{n} \| \boldsymbol{\phi}_i(t) \|^2 \le c_{12}\alpha^2$$

for some positive constants c_{11} and c_{12}. The case of time-varying β is a future research topic by the authors.

Remark 3. It is of interest to investigate if it is possible to extend the above results to any weighted consensus. Obviously, instead of Eq. (66) we should then have $p^T x(t) = p^T x(0)$, $\forall t \ge 0$, where p is a predefined vector specifying the desired weighting. This in turn implies that instead of Eq. (10), the coupling matrix $W(t)$ should satisfy $p^T W(t) = p^T$, $\forall t \ge 0$. At this point it is not clear how to select the appropriate cost function and the parameter estimator, different than those given by Eqs. (11), and (15), (16), so that $p^T W(t) = p^T$, $t \ge 0$, for a vector p specified by the designer.

Remark 4. Let us point out that the concept of adaptive weights (coupling parameters) in consensus algorithm has been employed in [28] where the authors proposed interesting consensus algorithm for distributed sensor fusion. The weight matrix is updated by using the steepest descent algorithm. The reference signal used in the algorithm is generated by specially designed linear predictor. In our case the weights are generated by using normalized gradient type algorithms, and the reference signal $\overline{x}_i(t+1)$ of the i-th agent is equal to the average of states of its neighbors.

4. Connection between Eq. (1) and frequency synchronization model

Building on the work by Winfree [17], in 1975 Kuramoto proposed his celebrated model describing a collective synchronization phenomenon [18]. This model is represented by $N \ge 2$ coupled oscillators whose dynamics are given by

$$\dot{\delta}_i(\tau) = \Omega_i + \sum_{j=1}^{N} \Gamma_{ij}\big(\delta_j(\tau) - \delta_i(\tau)\big), \quad i = 1, 2, \ldots, N \tag{68}$$

with $\delta_i(\tau)$ being the phase of the i-th oscillator, Ω_i is its natural frequency, and the interacting function $\Gamma_{ij}(\cdot)$ describe the coupling between the i-th and j-th oscillator. Kuramoto analyzed the following interacting functions:

$$\Gamma_{ij}\left(\delta_j(\tau)-\delta_i(\tau)\right)=\frac{K}{N}\sin\left(\delta_j(\tau)-\delta_i(\tau)\right) \tag{69}$$

where K defines the coupling strength among oscillators. It is not difficult to see that for the case of linear coupling the discrete time version of Eq. (68) is similar to the model given by Eqs. (1) and (2). Let $\Gamma_{ij}(z)=(\theta_{ij}/T_s)z$, and

$$\dot{\delta}_i(\tau)|_{\tau=tT_s}=\frac{\delta_i((t+1)T_s)-\delta_i(tT_s)}{T_s},\quad t=0,1,2,\ldots,$$

where T_s is the sampling interval. Then from Eq. (68) we can derive

$$x_i(t+1)=\Omega_i+\sum_{j=1}^{N}\theta_{ij}\left(\delta_j(t)-\delta_i(t)\right) \tag{70}$$

$$x_i(t+1)=\frac{\delta_i(t+1)-\delta_i(t)}{T_s},\quad t=0,1,\ldots, \tag{71}$$

where $x_i(t+1)$ is the normalized frequency of the i-th oscillator, while Ω_i and $\delta_i(t)$ have the same meaning as in Eq. (68). When denoting signals here, the constant T_s has been omitted, i.e., $\delta_i(tT_s)=\delta_i(t)$. It is obvious that Eq. (71) implies

$$x_i(t+1)-x_i(t)=\sum_{j=1}^{N}\theta_{ij}\left(x_j(t)-x_i(t)\right) \tag{72}$$

which is the model described by Eqs. (1) and (2). If we assume that in Eq. (70) $\delta_i(t)=0$ for $t<0$, then $x_i(0)=\Omega_i$, and model (72) together with the algorithm defined by Eqs. (14)–(17) provides the intended frequency synchronization, i.e., $\lim_{t\to\infty}x_i(t)=\left(\sum_{i=1}^{N}\Omega_i\right)/N$.

5. Simulation examples

Example 1. Consider network of six agents where undirected graph topology is defined by the following adjacent matrix:

$$\mathbf{A}_d=\begin{bmatrix}0&1&0&1&0&1\\1&0&1&0&0&0\\0&1&0&0&1&1\\1&0&0&0&1&0\\0&0&1&1&0&1\\1&0&1&0&1&0\end{bmatrix} \tag{73}$$

Here $\mathbf{A}_d(i,j)=1$ indicates that there is direct communication link between agent i and agent j. In Eq. (1) we set $\beta=-0.78$, and the initial states are selected as follows, $x_i(0)=2i,\ i=1,\ldots,6$. The step size μ_i in Eq. (15) is set to $\mu_i=1.5,\ 1\le i\le 6$. Matrix $\mathbf{A}_d$ shows that the first agent is connected with the 2nd, 4th, and 6th agents. Hence, vector $\boldsymbol{\theta}_1(t)$ in Eq. (6) has three nonzero components, $\theta_{12}(t)$, $\theta_{14}(t)$, and $\theta_{16}(t)$, whose evolution in time is depicted in Fig. 1. Fig. 2 shows that all agent states $x_i(t),\ i=1,\ldots,6$ converge toward consensus value equal to $(x_1(0)+\cdots+x_6(0))/6=7$.

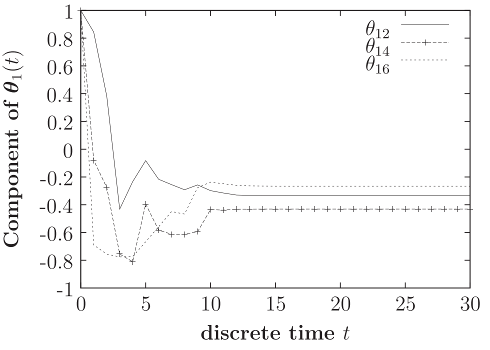

Fig. 1. Evolution of the parameter vector $\boldsymbol{\theta}_1(t)$ for Example 1.

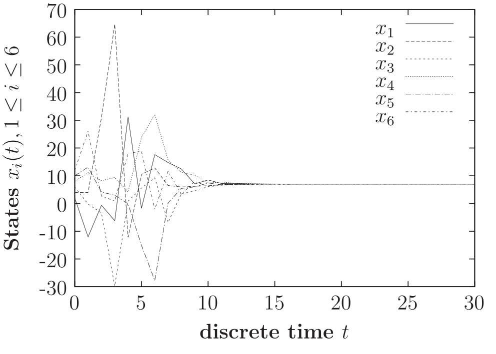

Fig. 2. The convergence of the network states for Example 1.

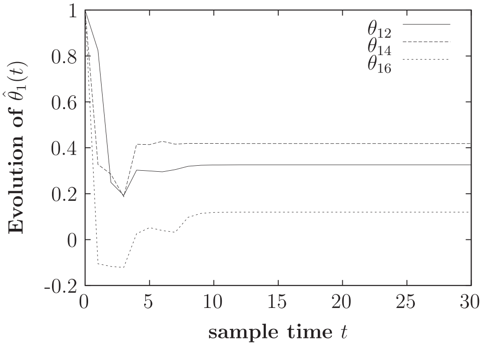

Fig. 3. Evolution of the parameter vector $\boldsymbol{\theta}_1(t)$ for Example 2.

Example 2. Consider the network from the previous example with $\beta(t) = 0.65 + 0.3\cos((2\pi/200)t)$, $t \geq 0$. In Eq. (15) we set $\mu_i = 1$ and use the same initial states $x_i(0)$ as in the previous example. Fig. 3 depicts the tuning of the vector $\boldsymbol{\theta}_1(t)$. Fig. 4 shows that despite time variations of $\beta(t)$, all agent states synchronize to the consensus value $x_c = 7$. The parameter estimator

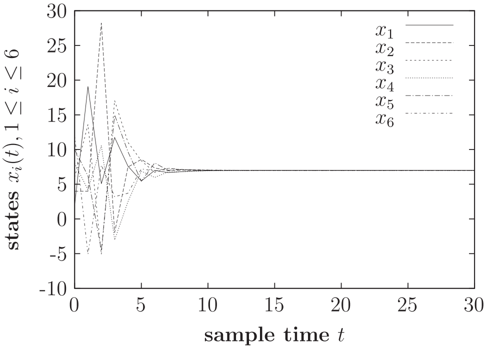

Fig. 4. The convergence of the network states for Example 2.

(14)–(17) is robust with respect to uncertainty of time-varying $\beta(t)$. From Theorem 1 it follows that as far as the size of μ_i, Assumption A2 is a sufficient condition (see relations (52) and (53)). Simulation experiments indicate that Assumption 2 is not a necessary condition. As a future research topic it is of interest to examine robustness of the tuning algorithm with respect to the rate of change of $\beta(t)$, and the interplay between $\beta(t)$ and the algorithm step size μ_i.

The unique feature of the proposed algorithm as compared to consensus protocols discussed in [16] is that the algorithm (15)–(17) provides average consensus in case of uncertain agent dynamics. Methods discussed in [16] assume that the agent dynamics is known. As far as the computational complexity, each agent has to run parameter estimator given by Eqs. (15)–(17) of similar complexity to a standard gradient based adaptive controller [25]. In other words, the i-th agent estimates N_i parameters, where N_i is the number of its neighbors, thus making the proposed algorithm to have the same complexity as the normalized least-mean-square (LMS) based adaptive filter of $N_i - 1$ order (with N_i taps).

6. Conclusions

In this paper we proposed a distributed averaging rule where each node of a network locally tunes its coupling parameters by using NGA recursion. It is shown that the coupling parameter sequence converges, and all agent states asymptotically reach consensus equal to the average of initial state values. As a future research topic, it is of interest to examine the behavior of the proposed algorithm in case of time-delay in information exchange between agents, time-varying network topologies, quantization errors, data-packets drops, and noisy measurements. Of particular interest would be to cast the considered consensus problem in the framework presented in [29] in case of state time delay, or [30] in case of feedback design based on quantized measurements. The authors are exploring the possibility of studying the robustness of the proposed algorithm in case of unreliable communication links by using tools developed in [31,32]. In order to reduce the amount of inter-agent communications and lower the frequency of weight updates, it is important to extend the derived results to the case of event-triggered consensus protocols by using the method developed in [33].

Appendix

Kronecker Lemma. *Assume that*

(i) $\sum_{k=1}^{t} a(k)$ *converges*

(ii) $\{r(t)\}, \quad t \geq 0$ *is a nondecreasing sequence*

(iii) $\lim_{t\to\infty} r(t) = \infty.$

Then

$$\lim_{t\to\infty} \frac{1}{r(t)} \sum_{k=1}^{t} r(k)a(k) = 0.$$

Proof. Proof of this lemma can be found in [25, p. 503]. □

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