

Distributed self-tuning consensus and synchronization



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HIGHLIGHTS

- New algorithm for distributed self-tuning synchronization of multi-agent systems.
- Error between velocity of an agent and the average of its neighbors is minimized
- Algorithm generates nonnegative and primitive inter-agent coupling matrix.
- The agent velocities converge toward same constant value.
- Velocities converge sufficiently fast so that distances between agents are bounded.

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ABSTRACT

The problem of self-tuning of coupling parameters in multi-agent systems is considered. Agent dynamics are described by a discrete-time double integrator with unknown input gain. Each agent locally tunes the strength of interaction with neighboring agents by using a normalized gradient algorithm (NGA). The tuning algorithm minimizes the square of the error between an individual agent's state (velocity) and the one step delayed average of its own state and the states of its neighbors. Assuming that the network graph is strongly connected, it is proved that the sequence of coupling parameters is convergent and all velocities converge toward the same constant value.

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1. Introduction

Synchronization processes represent a form of emergence in a population of networked systems. This intriguing phenomenon of collective behavior is observed in natural and man made systems in biology, chemistry, physics and engineering, as well as in the arts and social contexts. Winfree [1] assumed that a rhythmical coherent activity of a group can be modeled by a population of self-sustained and interacting oscillatory elements. One of the most popular models was proposed by Kuramoto [2] who considered a collection of limit-cycle oscillators each running at a different natural frequency, and coupled via a sine function of their phase differences. Generally speaking, Kuramoto oscillators synchronize when individual frequencies lock onto some common value. A comprehensive list of references on the subject of synchronization in oscillatory networks can be found in recent surveys [3–5]. Closely related to and often overlapping with synchronization is the so called consensus problem. As stated in [6], a group of interacting

dynamical systems (networked agents) achieves consensus when agreement is reached with respect to a certain variable that depends on the state of all of the agents. One of the first formal considerations of consensus describes how a group of individuals might reach agreement on a common probability distribution by pooling their individual opinions [7]. The work by Vicsek et al. [8] can be considered a motivational paper for many results in the area of consensus and presents a simple model of autonomous agents all moving in a plane with the same speed and different headings. Each agent adjusts its heading based on the average of the neighbors' headings including its own. Jadbabaie et al. [9] present a formal analysis for a distributed coordination model proposed in [8]. One of the first analytically rigorous formulations and treatments of consensus can be found in [9,10]. In the last fifteen years a large number of interesting results covering a variety of consensus aspects have been published. Topics such as distributed optimization and task assignments, coordination in swarms and flock formation, sensor fusion, and distributed estimation and control, have been extensively studied. A large number of references are given in survey papers [6,11–13], as well as in recently published research monographs [14–16]. These references consider a diverse set of issues such as the presence of noise and delay in communication links between agents, time varying topologies, asynchronous

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updating of agent states, quantization effects, and nonlinear dynamics. The following paragraphs provide a brief review of recent approaches to adaptive consensus.

Recent work on adaptive consensus: One of the earlier works on adaptive synchronization in dynamical networks is [17]. The authors assume that the synchronous solution of the overall autonomous network is known and is global information to be tracked by individual agents as a reference trajectory. The local control input is proportional to the error between the agent state and the reference trajectory. The authors prove that this error converges to zero. Similar results are presented in [18]. The assumption that the synchronous solution is known or that it is bounded is restrictive. In [19] the proof of global synchronization uses a circular condition related to adaptively changing coupling gains. This reference requires that a certain matrix dependent on the coupling gains is negative semidefinite at all times $t > 0$. In [20] the problem of steering a group of agents to a predefined reference velocity is considered. The reference velocity is known to the leader. It is also assumed that the reference velocity model is linear with respect to unknown parameters and known base functions are available to each agent. A decentralized adaptive design is proposed by incorporating relative position and relative velocity feedback. In [21,22] the authors consider the problem where every agent has to track a known or estimated leader trajectory. The agent dynamics are linear with respect to both the unknown parameters and the known basis functions. The leader trajectory is global information. The control signals are adaptive with respect to unknown parameters of agent dynamics. The inter-agent coupling parameter is a nonadaptive predefined constant whose value is global information and is the same for all agents. The local input signal resembles a high-frequency gain feedback in decentralized control methods. In [23] the authors analyze the undirected graph topology, and assume that the high frequency gain (parameter multiplying agent input signal) is known, and it is the same for all agents. They proved the interesting result that each agent state converges toward the average of its neighbors' states. In [24] the consensus problem with a general linear model and Lipschitz nonlinear dynamics is considered. The authors analyze an undirected graph and assume that the linear dynamics are known. The proposed consensus protocol can be implemented in a distributed fashion. A continuous time consensus problem of second order systems governed by a directed graph is considered in [25]. The authors show that the error between any two agent positions converges to zero. They also show that in case of absolute velocity damping all velocities converge to zero, while in the case of relative velocity damping the difference between agent velocities converges to zero. In recent work by Chen et al. [26] continuous time adaptive consensus with unknown identical control directions is considered. The authors analyze an undirected graph and show that the difference between agent states tends to zero.

In [27] the authors consider the finite time leader following problem of multi-agent systems whose dynamics is linear with respect to unknown parameters and known basis functions. Similarly as in [21,23] the inter-agent coupling parameters are nonadaptive, pre-calculated and same for each agent. The leader following is achieved in a finite time. In [28] the consensus problem of networked mechanical systems with time-varying delay and jointly connected topologies is considered. Similarly as in [21,22,27] it is assumed that the high-frequency gain is known, and the inter-agent coupling term in the consensus protocol is non-adaptive with a fixed gain whose value is the same for all agents. In [29] the authors investigate the cooperative control of networked agents with unknown control directions. Assuming undirected graph topology they propose interesting Nussbaum type adaptive controller, and showed that all signals are bounded. They also prove that the difference between any two agent states

asymptotically goes to zero. Note that this statement does not imply that all agent states have finite limit.

Contribution and organization: Here we consider a network of heterogeneous agents whose dynamics are described by a double integrator discrete time model with input gain of unknown magnitude. Motivated by the evolution of flocks in biology, or the engineering problem of control of formations of unmanned mobile agents, we set out to find an algorithm for each agent to locally tune the inter-agent coupling parameter so that (i) all agent velocities converge to the same value, and (ii) the distance between any two agents converges to a finite limit without using a predefined reference (velocity or position) trajectory.

The proposed algorithm is a normalized gradient recursion based on minimizing the square of the error between an agent state and the one step delayed average of the state's of its neighbors. In the following we list our contribution relative to the recent work of Junmin and Xudong [29]. Ref. [29] considers continuous-time adaptive consensus; we analyze discrete-time adaptive consensus. In [29] the consensus algorithm is constructed based on the Lyapunov function argument while our algorithm is a normalized gradient scheme derived by minimizing a certain quadratic cost function and it is different than the algorithm in [29]. We consider double integrator discrete-time dynamics, while in [29] a single integrator continuous-time system is discussed. Ref. [29] analyzes an undirected graph while we consider a more general directed graph topology. In [29] it is shown that agent states are bounded and the error between any two agents states goes to zero. Note that this statement does not imply that all agent states have a limit. We prove that all agent states converge to the same value. In [29] it is shown that the coupling parameters are bounded, not necessarily convergent functions. We prove that the coupling parameters are convergent sequences. In addition we show that the distance between any two members of the group converges toward a finite limit.

The paper is organized as follows. Section 2 presents the problem formulation. Section 3 presents the proposed algorithm. Analysis of the algorithm is presented in Section 4. A simulation example is given in Section 5. We use the following notation: \Re denotes the set of real numbers; the superscript T denotes the transpose of a matrix; $\rho(A)$ denotes the spectral radius of matrix A ; $\|x\|$ is the Euclidean norm of vector x , and $\text{sgn}(a)$ is the sign function of a real number a . Furthermore, ℓ is used to denote a vector with all entries equal to one, i.e., $\ell^T = [1, 1, \dots, 1]$. When performing majorizations and in certain upper bounds, c_i , $i = 1, 2, \dots$ is used to denote nonnegative constants whose values are unimportant.

2. Problem statement

Consider a cooperative group of N agents where the dynamics of the i th agent are described by the following discrete time system

$$x_i(t+1) = x_i(t) + v_i(t) \quad (1)$$

$$v_i(t+1) = v_i(t) + \beta_i u_i(t), \quad i = 1, \dots, N \quad (2)$$

where time $t \geq 0$ takes on nonnegative integer values, $x_i(t) \in \Re$ and $v_i(t) \in \Re$ are the position and velocity respectively, while $u_i(t) \in \Re$ is the control signal or consensus protocol of the agent. In (2) $\beta_i \in \Re$ is an unknown input gain. The model defined by Eq. (2) can be thought of as a discrete time version of a kinematic model

$$\frac{d}{d\tau} v_i(\tau) = \frac{1}{m_i} u_i(\tau), \quad \tau \geq 0,$$

for $i = 1, \dots, N$, where $v_i(\tau)$ is velocity and $u_i(\tau)$ is driving force of the i th agent respectively, while m_i is its mass. Then parameter β_i in (2) can be interpreted as an inverse of m_i . Inspired by

flocking behavior, model (1), (2) is simple and effective and can be a useful tool in explaining synchronous behavior in biological systems or used in applications such as robot formation control. The communication topology of the above network of agents is represented by a directed graph $G = (\mathcal{V}, \mathcal{E})$ with the set of nodes $\mathcal{V} = \{1, 2, \dots, N\}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges or communication links. The node i represents the agent i , and ordered pairs (i, j) $i \neq j$ denote edges, where $(i, j) \in \mathcal{E}$ if and only if the agent i can directly receive information from the j th agent. The set of neighbors of node i is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} \mid (i, j) \in \mathcal{E}\}$.

It is plausible to assume that in natural phenomena such as flocks, each agent adjusts its velocity so that it is as close as possible to the average of the velocities of its neighbors, i.e. $u_i(t)$ is proportional to the local velocity mismatch defined by

$$\varphi_i(t) = \bar{v}_i(t) - v_i(t) \quad (3)$$

where

$$\bar{v}_i(t) = \frac{1}{1 + N_i} \sum_{j \in \mathcal{N}'_i} v_j(t) \quad (4)$$

where N_i is the cardinality of \mathcal{N}_i , and $\mathcal{N}'_i = \mathcal{N}_i \cup \{i\}$. Thus

$$u_i(t) = \theta_i(t) \varphi_i(t), \quad (5)$$

where $\theta_i(t) \in \mathfrak{R}$ is the coupling parameter. In the next section we develop a tuning rule for $\theta_i(t)$ so that all agent velocities converge toward the same value. By substituting (5) in (2) we arrive at the following evolution of $v_i(t)$,

$$v_i(t+1) = v_i(t) + \beta_i \theta_i(t) \varphi_i(t), \quad 1 \leq i \leq N. \quad (6)$$

Now that we have derived model (6) we can relate multi-agent velocity consensus to frequency synchronization problems in linearly phase coupled oscillators described by

$$\dot{x}_i(\tau) = \Omega_i + K_i \frac{1}{1 + N_i} \sum_{i \in \mathcal{N}'_i} (x_j(\tau) - x_i(\tau)), \quad i \in \mathcal{V} \quad (7)$$

where $x_i(\tau)$ is the phase of the i th oscillator, Ω_i is its natural frequency and K_i is the coupling gain. Let frequency $\dot{x}_i(\tau)$ at time $\tau = tT_s$, $t = 0, 1, 2, \dots$, be approximated by

$$\left. \frac{d}{d\tau} x_i(\tau) \right|_{\tau=tT_s} = \frac{x_i((t+1)T_s) - x_i(tT_s)}{T_s}$$

where T_s is the sampling interval. Then from (7) we can write

$$v_i(t+1) = \Omega_i T_s + \theta_i \psi_i(t) \quad (8)$$

$$v_i(t+1) = x_i(t+1) - x_i(t), \quad v_i(0) = \Omega_i T_s \quad (9)$$

where

$$\psi_i(t) = \frac{1}{1 + N_i} \sum_{j \in \mathcal{N}'_i} (x_j(t) - x_i(t)) \quad (10)$$

and $\theta_i = K_i T_s$. The initial condition $v_i(0)$ is determined so that $v_i(t) = 0$ and $x_i(t) = 0$ for all $t < 0$. Note that for the sake of simpler notation the constant T_s has been omitted in signal arguments, i.e., $x_i(tT_s) = x_i(t)$, $t \geq 0$. Obviously we can think of $v_i(t)$ as the “normalized frequency” at a discrete time t . Since from (3), (4) and (11), $\varphi_i(t) = \psi_i(t) - \psi_i(t-1)$ Eq. (8) can be written as $v_i(t+1) - v_i(t) = \theta_i \varphi_i(t)$, $v_i(0) = \Omega_i T_s$, $i \in \mathcal{V}$, which is the same as consensus model (6).

3. Normalized gradient algorithm for self-tuning consensus

In this section we develop the algorithm for tuning coupling parameters $\theta_i(t)$, $t \geq 0$, $i \in \mathcal{V}$. Observe that Eq. (6) can be written in compact form as

$$v(t+1) = W(t)v(t) \quad (11)$$

where $v(t)$ is given by

$$v(t)^T = [v_1(t), \dots, v_N(t)] \quad (12)$$

and $W(t)$ is a $N \times N$ matrix of coupling weights defined as follows:

$$W(t) = [w_{ij}(t)], \quad w_{ij}(t) = \begin{cases} \beta_i \theta_i(t) \frac{1}{1 + N_i}, & j \in \mathcal{N}_i, \\ 1 - \beta_i \theta_i(t) \frac{N_i}{1 + N_i}, & j = i \\ 0, & \text{otherwise.} \end{cases} \quad (13)$$

In most literature on consensus theory, $W(t)$ is a non-negative row stochastic matrix. We now develop an algorithm for agent i to locally tune coupling $\theta_i(t)$, $i \in \mathcal{V}$ so that $W(t)$ is a nonnegative matrix that will guarantee $v_i(t) \rightarrow v_c$ as $t \rightarrow \infty$, for some finite v_c . Agent $i \in \mathcal{V}$ tunes coupling parameter $\theta_i(t)$ so that the following local cost function is minimized.

$$J_i(\theta_i) = \frac{1}{2} (v_i(t+1) - \bar{v}_i(t+1))^2, \quad i \in \mathcal{V} \quad (14)$$

where $\bar{v}_i(t+1)$ represents the one step delayed weighted average of the i th agents neighbors' velocities, including its own velocity i.e.,

$$\bar{v}_i(t+1) = \sum_{j=1}^N m_{ij} v_j(t) \quad (15)$$

where

$$m_{ij} = \begin{cases} \frac{1 - \alpha_i}{1 + N_i}, & j \in \mathcal{N}_i, 0 \leq \alpha_i < 1 \\ \alpha_i + \frac{1 - \alpha_i}{1 + N_i}, & j = i \\ 0, & \text{otherwise.} \end{cases} \quad (16)$$

By virtue of the fact that from (6) and (14) the gradient of $J_i(\theta_i)$ with respect to $\theta_i(t)$ is

$$\frac{\partial J_i(\theta_i)}{\partial \theta_i(t)} = (v_i(t+1) - \bar{v}_i(t+1)) \beta_i \varphi_i(t),$$

$\theta_i(t)$ can be tuned by the following recursive procedure

$$\theta_i(t+1) = \theta_i(t) - \beta_i \varphi_i(t) (v_i(t+1) - \bar{v}_i(t+1)), \quad i \in \mathcal{V}. \quad (17)$$

However, since β_i is unknown, instead of (17) agent i can use the following normalized gradient algorithm

$$\theta_i(t+1) = \theta_i(t) - \frac{\mu_i}{r_i(t)} \text{sgn}(\beta_i) \varphi_i(t) e_i(t+1), \quad i \in \mathcal{V} \quad (18)$$

where it is assumed that $\text{sgn}(\beta_i)$, the sign of β_i is known, $\mu_i > 0$ is the algorithm step size, $\varphi_i(t)$ is the local velocity mismatch defined by (3), $e_i(t+1)$ is the cost function error given by

$$e_i(t+1) = v_i(t+1) - \bar{v}_i(t+1) \quad (19)$$

with $\bar{v}_i(t+1)$ defined by (15), and $r_i(t)$ being the gradient normalizer given by

$$r_i(t) = 1 + \varphi_i(t)^2, \quad 1 \leq i \leq N. \quad (20)$$

The role of $r_i(t)$ will become clear when we analyze global stability of (18). Recursion (18) starts with some finite initial condition

$\theta_i(0)$. Because of its simplicity, Eqs. (3), (6) and (18) can be viewed as an effective model to explain how the process of tuning the interaction strengths among agents, lead toward synchronous behavior. It is obvious from (15) that when calculating $\bar{v}_i(t+1)$ agent i assigns more weight to its value than to any individual neighbor. When in (16) $\alpha_i = 0$, the average $\bar{v}_i(t+1)$ becomes equal to the uniformly weighted average. In the next section we show that $W(t)$ in Eq. (13) will be a nonnegative matrix if in (16) $\alpha_i \geq 1/2$, $\forall i \in \mathcal{V}$. It is not difficult to see that both averages $\bar{v}_i(t)$ and $\bar{v}_i(t+1)$ are related as follows:

$$\bar{v}_i(t+1) = \alpha_i v_i(t) + (1 - \alpha_i) \bar{v}_i(t). \quad (21)$$

For the sake of clarity of the analysis in the next section, we introduce the following definitions regarding leaderless consensus.

Definition 1. A multi-agent system defined by (6) and (18) achieves wide-sense consensus if

$$\lim_{t \rightarrow \infty} (v_i(t) - v_j(t)) = 0, \quad \forall i, j \in \mathcal{V}. \quad (22)$$

Obviously (22) does not imply that $\{v_i(t), t \geq 0$ is a bounded or convergent sequence. For example, if we consider the following sequences: $v_i(t) = \log(t+i)$, or $v_i(t) = (-1)^t + \frac{i}{t}$, or $v_i(t) = \cos(\log(t+i))$, $i \in \mathcal{V}$, we see that (22) holds for all of them and yet $\lim_{t \rightarrow \infty} v_i(t)$ does not exist. For this reason we define a stronger type of consensus.

Definition 2. A multi-agent system achieves strict-sense consensus if (22) holds and $\lim_{t \rightarrow \infty} v_i(t) = v_c$, $\forall i \in \mathcal{V}$ for some finite x_c .

In the next section we demonstrate that the coupling parameters $\{\theta_i(t)\}$, $t \geq 0$, $i \in \mathcal{V}$ are convergent sequences. We also prove that $\lim_{t \rightarrow \infty} v_i(t)$ exists, $\forall i \in \mathcal{V}$, thus our algorithm achieves strict sense consensus as previously defined. In [29] it is shown that the states are bounded, but their convergence is not established. Another contribution of our paper is the proof that the distance $x_i(t) - x_j(t)$ between any two members of the group converges toward a finite limit as $t \rightarrow \infty$. In case of adaptive consensus, this is not a straightforward conclusion. This can be seen from Eq. (1) which gives $x_i(t) - x_j(t) = x_i(0) - x_j(0) + \sum_{k=0}^t (v_i(k) - v_j(k))$, from where it is obvious that $x_i(t) - x_j(t)$ may diverge despite the fact that $v_i(t)$, $i \in \mathcal{V}$ is a convergent sequence. The distance will converge only if $v_i(t) - v_j(t)$ converges to zero sufficiently fast.

4. Stability and convergence of the self-tuning consensus

We first show that sequences $\{e_i(t)\}$ and $\{\varphi_i(t)\}$, $t \geq 0$ generated by the algorithm (18)–(20) along with system dynamics (6) have finite total energies for all finite initial conditions $\theta_i(0)$, $x_i(0)$, $v_i(0)$, $1 \leq i \leq N$. In this paper we make use of the following assumptions.

Assumption A1. The underlying directed graph G is strongly connected.

Assumption A2. The sign of β_i and the upper bound $\beta_{i\max}$ of $|\beta_i|$ are known to agent i . The step size μ_i in (14) satisfies $\mu_i < 2/\beta_{i\max}$, for all $i = 1, \dots, N$.

Define vector

$$e(t)^\top = [e_1(t), \dots, e_N(t)]. \quad (23)$$

Then from (12), (15) and (19) we can write

$$v(t+1) = Mv(t) + e(t+1) \quad (24)$$

where M is the $N \times N$ matrix given by $M = [m_{ij}]$ with elements m_{ij} defined by (16). By using the fact that $\sum_{j=1}^N m_{ij} = 1$ for all $i \in \mathcal{V}$, we conclude that $\lambda_1 = 1$ is an eigenvalue of M with the corresponding right eigenvector $\ell = (1, \dots, 1)^\top$. Since M is a stochastic matrix, λ_1 is its maximal eigenvalue [30, p. 83]. By virtue of the fact that the underlying graph G is strongly connected, nonnegative matrix M is irreducible implying that λ_1 is an algebraically simple eigenvalue [31, Theorem 8.4.4, p. 508]. It is worth mentioning that this statement is also known as the Perron–Frobenius theorem, and it is a generalization of Perron's results for positive matrices to nonnegative matrices. Furthermore, if the nonnegative matrix is irreducible and any main diagonal element is positive, such matrix must be primitive [31, see Theorems 8.5.2 and 8.5.10, p. 516]. This implies that M has only one eigenvalue of maximum modulus.

Let $\lambda_1 \geq |\lambda_2| \geq \dots \geq |\lambda_N|$ be the ordered eigenvalues of M . Since the maximal eigenvalue $\lambda_1 = 1$ is simple, we have $|\lambda_i| < 1$ for $2 \leq i \leq N$. Let y_M be the left eigenvector of M corresponding to $\lambda_1 = 1$, and normalized so that $\ell^\top y_M = 1$. Based on the above discussion matrix M can be decomposed as follows:

$$M = M_1 + \ell y_M^\top, \quad y_M^\top \ell = 1 \quad (25)$$

where $M_1 \ell = 0$, $y_M^\top M_1 = 0$ and the spectral radius $\rho(M_1) < 1$. Then from (24) we can derive

$$v(t+1) = M_1 v(t) + \ell y_M^\top v(t) + e(t+1). \quad (26)$$

Define

$$L(q^{-1}) = (I - q^{-1} M_1)^{-1} \quad (27)$$

where q^{-1} is the unit delay operator. Since $\rho(M_1) < 1$ and $M_1 \ell = 0$ we have $(I - q^{-1} M_1)^{-1} \ell = (I + \sum_{k=1}^{\infty} M_1^k) \ell = \ell$. Then (26) gives

$$v(t+1) = \ell y_M^\top v(t) + L(q^{-1}) e(t+1) \quad (28)$$

with $L(q^{-1})$ being a stable operator due to the fact that $\rho(M_1) < 1$.

The following technical result is needed for future reference.

Lemma 1. Let Assumption A1 hold. Then for all $n \geq 0$ and $i \in \mathcal{V}$,

$$\sum_{t=0}^n \|\phi_i(t)\|^2 \leq c_1 + c_2 \sum_{t=0}^n \|e(t)\|^2 \quad (29)$$

where

$$\phi_i(t) = v(t) - v_i(t) \ell, \quad i \in \mathcal{V}. \quad (30)$$

Proof. By using the fact that $y_M^\top \ell = 1$, from (30) and (28) we have

$$\begin{aligned} \phi_i(t+1) &= \ell y_M^\top \phi_i(t) + \ell (v_i(t) - v_i(t+1)) \\ &\quad + L(q^{-1}) e(t+1). \end{aligned} \quad (31)$$

Since from (16) $m_{ii} = 1 - \sum_{j \in \mathcal{N}_i} m_{ij}$, it is not difficult to see that Eq. (15) can be written as follows:

$$\bar{v}_i(t+1) = v_i(t) + a_i^\top \phi_i(t) \quad (32)$$

where $\phi_i(t)$ is defined by (30) and

$$a_i^\top = [a_{i1}, \dots, a_{iN}], \quad a_{ij} = \begin{cases} \frac{1 - \alpha_i}{1 + N_i}, & j \in \mathcal{N}_i, \\ 0, & \text{otherwise.} \end{cases} \quad (33)$$

Substituting (32) in (19) yields

$$e_i(t+1) = v_i(t+1) - v_i(t) - a_i^\top \phi_i(t). \quad (34)$$

Then inserting $v_i(t+1) - v_i(t)$ from (34) in (31) gives $\phi_i(t+1) = Q_i \phi_i(t) - e_i(t+1)\ell + L(q^{-1})e(t+1)$, or

$$\phi_i(t+1) = (I - q^{-1}Q_i)^{-1}[L(q^{-1})e(t+1) - e_i(t+1)\ell] \quad (35)$$

where $Q_i = \ell(y_M - a_i)^T$, $1 \leq i \leq N$. Note that Q_i is a rank one matrix and its only nonzero eigenvalue is $\rho_1 = \ell^T(y_M - a_i)$. Since $\ell^T y_M = 1$ and from (33) $\ell^T a_i = \frac{1-\alpha_i}{1+N_i} N_i$, we have $\rho_1 = 1 - \ell^T a_i = \frac{1-\alpha_i N_i}{1+N_i} < 1$, where we have used the fact that $\alpha_i < 1$. Hence, all eigenvalues of Q_i are strictly inside the unit circle and consequently $(I - q^{-1}Q_i)^{-1}$ is a stable operator. Then from (35) one can derive

$$\begin{aligned} & \sum_{t=0}^n \|\phi_i(t+1)\|^2 \\ & \leq c_3 + c_4 \sum_{t=0}^n e_i(t+1)^2 + c_5 \sum_{t=0}^n \|L(q^{-1})e(t+1)\|^2 \end{aligned}$$

where the nonnegative constant c_3 is used to account for the nonzero initial conditions. By virtue of the fact that $L(q^{-1})$ is a stable operator, the previous relation implies

$$\sum_{t=0}^n \|\phi_i(t+1)\|^2 \leq c_3 + c_4 \sum_{t=0}^n e_i(t+1)^2 + c_6 \sum_{t=0}^n \|e(t+1)\|^2,$$

for all $n \geq 0$, $i \in \mathcal{V}$. Since $e_i(t)$ is the i th component of $e(t)$, the second term on the RHS of the last relation can be absorbed by the third term to give (29). Thus the lemma is proved. \square

The following lemma states that the equilibrium state $(e_i(t), \varphi_i(t)) = (0, 0)$ of the system dynamics defined by (6), (18)–(20) is globally asymptotically stable in the sense that $e_i(t) \rightarrow 0$, $(v_i(t) - v_j(t)) \rightarrow 0$, and consequently $\varphi_i(t) \rightarrow 0$ as $t \rightarrow \infty$, $i \in \mathcal{V}$, $j \in \mathcal{V}$. By global we mean that the above claim holds for all initial conditions $\theta_i(0)$, and $v_i(0)$, $i \in \mathcal{V}$. As a matter of fact we prove the more general statement that the sequences $\{e_i(t)\}$, $\{(v_i(t) - v_j(t))\}$, $i \in \mathcal{V}$, $j \in \mathcal{V}$ have finite energies. In addition we show that $\lim_{t \rightarrow \infty} \theta_i(t)$ exists $\forall i \in \mathcal{V}$.

Lemma 2. Let Assumptions A1 and A2 hold. Then for all initial conditions $v_i(0)$, $x_i(0)$ and $\theta_i(0)$ the algorithm defined by (6), (18)–(20) provides

$$(1) \sum_{t=0}^n \|e(t+1)\|^2 \leq c_7 < \infty, \quad \forall n \geq 0 \quad (36)$$

$$(2) \sum_{t=0}^n (v_i(t) - v_j(t))^2 \leq c_8 < \infty, \quad \forall n \geq 0, \forall i, j \in \mathcal{V} \quad (37)$$

$$(3) \sum_{t=0}^n \|v(t+1) - v(t)\|^2 < \infty, \quad \forall n \geq 0 \quad (38)$$

$$(4) \lim_{t \rightarrow \infty} \theta_i(t) = \bar{\theta}_i, \quad i \in \mathcal{V} \quad (39)$$

for some finite $\bar{\theta}_i$.

Proof. Define

$$\tilde{\theta}_i(t) = \beta_i \theta_i(t) - (1 - \alpha_i). \quad (40)$$

We first show that the error $e_i(t+1)$ defined by Eq. (19) can be expressed in terms of $\tilde{\theta}_i(t)$ as follows:

$$e_i(t+1) = \tilde{\theta}_i(t) \varphi_i(t), \quad i \in \mathcal{V} \quad (41)$$

where $\varphi_i(t)$ is given by (3). Note that by substituting (21) in (19) we obtain

$$e_i(t+1) = v_i(t+1) - \alpha_i v_i(t) - (1 - \alpha_i) \bar{v}_i(t). \quad (42)$$

Then substituting (2) into (42) yields

$$e_i(t+1) = \beta_i u_i(t) - (1 - \alpha_i) \varphi_i(t) \quad (43)$$

with $\varphi_i(t)$ defined by Eq. (3). Eq. (41) follows from Eqs. (5), (40), and (43). We now focus on the iterative scheme given by Eq. (18). After multiplying both sides of (18) with β_i and subtracting $1 - \alpha_i$ we can write

$$\tilde{\theta}_i(t+1) = \tilde{\theta}_i(t) - \frac{\mu_i}{r_i(t)} |\beta_i| \varphi_i(t) e_i(t+1). \quad (44)$$

Squaring both sides of the previous equation yields

$$\begin{aligned} \tilde{\theta}_i(t+1)^2 &= \tilde{\theta}_i(t)^2 - \frac{2\mu_i}{r_i(t)} |\beta_i| \tilde{\theta}_i(t) \varphi_i(t) e_i(t+1) \\ &\quad + (\mu_i \beta_i)^2 \frac{\varphi_i(t)^2}{r_i(t)} \frac{e_i(t+1)^2}{r_i(t)}. \end{aligned} \quad (45)$$

Since by (20) $\varphi_i(t)^2/r_i(t) \leq 1$, relations (41) and (45) imply

$$\tilde{\theta}_i(t+1)^2 \leq \tilde{\theta}_i(t)^2 - 2\mu_i |\beta_i| \left(1 - \frac{\mu_i |\beta_i|}{2}\right) \frac{e_i(t+1)^2}{r_i(t)}. \quad (46)$$

Define

$$\begin{aligned} V_i(t) &= \tilde{\theta}_i(t)^2 + 2\mu_i |\beta_i| \left(1 - \frac{\mu_i |\beta_i|}{2}\right) \sum_{k=0}^t \frac{e_i(k+1)^2}{r_i(k)}, \\ & i \in \mathcal{V}. \end{aligned} \quad (47)$$

Since by Assumption A2, $1 - \mu_i |\beta_i|/2 \geq 1 - \mu_i |\beta_{i\max}|/2 > 0$, we have $V_i(t) \geq 0$, $\forall t \geq 0$. Then from (46) and (47) it follows that $V_i(t+1) \leq V_i(t)$, i.e., $\{V_i(t)\}$ is a nonnegative non-increasing sequence satisfying $V_i(t) \leq V_i(0) < \infty$, $\forall t \geq 0$, $i \in \mathcal{V}$. Thus from (47) we can conclude that the following limit exists,

$$\lim_{t \rightarrow \infty} \sum_{k=0}^t \frac{e_i(k+1)^2}{r_i(k)} = c_9 < \infty, \quad 1 \leq i \leq N, \quad (48)$$

for all $t \geq 0$. Note that $V_i(t)$ defined by (47) represents a discrete-time Lyapunov function for the system described by (41) and (18). Let

$$r(t) = \sum_{i=1}^N \max_{0 \leq \tau \leq t} r_i(\tau). \quad (49)$$

The following is then obtained from (48) and (49)

$$\sum_{k=0}^t \frac{\|e(k+1)\|^2}{r(k)} \leq \sum_{i=1}^N \sum_{k=0}^t \frac{e_i(k+1)^2}{r_i(k)} \leq c_{10} < \infty. \quad (50)$$

If $\{r(t)\}$, $t \geq 0$ is a bounded sequence, then statement (36) follows directly from (50). If $r(t) \rightarrow \infty$ as $t \rightarrow \infty$, by Kronecker's lemma [32, p. 503] we conclude

$$\lim_{t \rightarrow \infty} \frac{1}{r(t)} \sum_{k=0}^t \|e(k+1)\|^2 = 0. \quad (51)$$

Note that Eq. (3) can be written in the form

$$\varphi_i(t) = b_i^T \phi_i(t) \quad (52)$$

where $\phi_i(t)$ is given by (30) and $b_i^T = [b_{i1}, \dots, b_{iN}]$, with

$$b_{ij} = \begin{cases} \frac{1}{1+N_i}, & j \in \mathcal{N}_i, \\ 0, & \text{otherwise.} \end{cases}$$

Then from (20), (52) and (29) we can derive

$$r(t) \leq \sum_{k=0}^t \sum_{i=1}^N \varphi(k)^2 \leq c_{11} \sum_{k=0}^t \|e(k+1)\|^2. \quad (53)$$

Statement (36) follows from (51) and (53). Observe that Lemma 1 and (36) yield $\sum_{t=0}^n \|\phi_i(t)\|^2 < \infty$, $\forall n \geq 0$, $i \in \mathcal{V}$. This relation together with (30) implies (37). Next we prove statement (39). Define

$$R_i(t+1) = \mu_i \sum_{k=0}^t \frac{1}{r_i(k)} \varphi_i(k) e_i(k+1), \quad R_i(0) = 0 \quad (54)$$

for $t \geq 0$. Since from (29), (36) and (52)

$$\sum_{k=0}^t \varphi(k)^2 \leq c_{12} < \infty, \quad \forall t \geq 0, \quad (55)$$

by using the Cauchy–Schwarz inequality, from (36) and (55) we can conclude

$$|R_i(t)| \leq \mu_i \left(\sum_{k=0}^t \varphi_i(k)^2 \right)^{1/2} \left(\sum_{k=0}^t e_i(k+1)^2 \right)^{1/2} \leq c_{13} < \infty. \quad (56)$$

Hence $R_i(t)$ is absolutely convergent and $R_i(t) \rightarrow \bar{R}_i$ for some finite \bar{R}_i . On the other hand, from (18) it follows that $\theta_i(t+1) + R_i(t+1)$ is time invariant, i.e., $\theta_i(t+1) + R_i(t+1) = \theta_i(t) + R_i(t) = \dots = \theta_i(0) + R_i(0)$ from where it follows that $\lim_{t \rightarrow \infty} \theta_i(t)$ exists. Statement (38) directly follows from (6), (39) and (55). Thus the lemma is proved. \square

Note that (37) is stronger than (22) and it implies (22). We now remark on the role of the normalizer $r_i(t)$ in (18). Assume for a moment that $r_i(t)$ is equal to one. Then if at some time instant, $\|\varphi_i(t)\|^2$ becomes too large, the third term on the RHS of (45) can dominate the second term, and $\bar{\theta}_i(t)^2$ may not be a non increasing function of time t . In combination with an appropriate choice of the step size μ_i , the normalizer $r_i(t)$ guarantees that the second term on the RHS of (45) is larger than the third term, and thus $\bar{\theta}_i(t+1)^2 \leq \bar{\theta}_i(t)^2$ for all $t \geq 0$ (see relation (46)).

Note that statements (37) and (38) do not imply that $\{v_i(t)\}$, $i \in \mathcal{V}$ is a convergent sequence. Next we analyze Eq. (11) and show that $\lim_{t \rightarrow \infty} v_i(t) = v_c$ for all $i \in \mathcal{V}$, and some finite v_c . It is important to observe that convergence toward a common velocity does not guarantee that the distance among agents $\{x_i(t) - x_j(t)\}$, $i, j \in \mathcal{V}$, $i \neq j$ is a bounded sequence. For example, if agent velocities evolve according to $v_i(t) = v_c + i/t$, $i \in \mathcal{V}$ from (1) we can conclude that the distance between the i th and j th agent is unbounded, i.e. $x_i(t) - x_j(t) = \Theta(\log t)$, where $\Theta(\log t)$ signifies that the distance goes to infinity as $\log(t)$. In this paper we demonstrate that $v_i(t)$, $i \in \mathcal{V}$ converges toward a common v_c sufficiently fast such that the distances between agents remain bounded. We now turn to Eqs. (11) and (13).

Observe that from (39) it follows $W(t) \rightarrow \bar{W}$ as $t \rightarrow \infty$, where

$$\bar{W} = [\bar{w}_{ij}], \quad \bar{w}_{ij} = \begin{cases} \beta_i \bar{\theta}_i \frac{1}{1 + N_i}, & j \in \mathcal{N}_i, \\ 1 - \beta_i \bar{\theta}_i \frac{N_i}{1 + N_i}, & j = i \\ 0, & \text{otherwise.} \end{cases} \quad (57)$$

Hence from (11) we can write

$$v(t+1) = \bar{W}v(t) + \tilde{W}v(t) \quad (58)$$

with $\tilde{W}(t) = W(t) - \bar{W} \rightarrow 0$ as $t \rightarrow \infty$. Before we show that all velocities converge toward the same value, we demonstrate that $W(t)$ and \bar{W} are nonnegative matrices. We prove this under the following constraint on the initial conditions $\theta_i(0)$, $i \in \mathcal{V}$.

Assumption A3. In (18), $\theta_i(0)$ is selected so that $\theta_i(0) = \theta_{0i} \operatorname{sgn}(\beta_i)$, $0 < \theta_{0i} < 2(1 - \alpha_i)/\beta_{i\max}$, $1/2 \leq \alpha_i < 1$.

Recall that α_i is a parameter defining the weight of $v_i(t)$ in the average $\bar{v}_i(t+1)$ given by Eq. (15). The previous assumption requires $\alpha_i \geq 0.5$. Then from (15) it is not difficult to see that the i th agent calculates $\bar{v}_i(t+1)$ by including its own value $v_i(t)$ in this average with the weight larger than 0.5. In the following we show that Assumption A3 is sufficient for the parameter estimator (18) to produce a $\theta_i(t)$ such that \bar{W} in Eqs. (57) and (58) is a nonnegative row stochastic matrix. (Recall that in (57), $\bar{\theta}_i = \lim_{t \rightarrow \infty} \theta_i(t)$.)

Observe that the previous assumption implies $0 < \beta_i \theta_i(0) = |\beta_i| \theta_{0i} < 2(1 - \alpha_i)$ or $|\beta_i \theta_i(0) - (1 - \alpha_i)| < (1 - \alpha_i)$. Then from (46) and (40) we derive

$$(\beta_i \theta_i(t) - (1 - \alpha_i))^2 \leq (\beta_i \theta_i(0) - (1 - \alpha_i))^2 < (1 - \alpha_i)^2. \quad (59)$$

Hence

$$0 < \beta_i \theta_i(t) < 2(1 - \alpha_i) \quad (60)$$

for all $t \geq 0$. By continuity, the same holds for $\bar{\theta}_i = \lim_{t \rightarrow \infty} \theta_i(t)$, i.e.

$$0 < \beta_i \bar{\theta}_i < 2(1 - \alpha_i), \quad 1 \leq i \leq N. \quad (61)$$

Thus \bar{W} in (57) is a nonnegative matrix. Since by construction it is a row stochastic matrix, $\lambda_1 = 1$ is its maximal eigenvalue [30, p. 83]. By the fact that the corresponding graph is strongly connected, \bar{W} is an irreducible matrix. Then by the Perron–Frobenius theorem for nonnegative matrices, $\lambda_1 = 1$ is an algebraically simple eigenvalue (Theorem 8.4.4, p. 508 in [31]). Since $w_{ii} = 1 - \beta_i \bar{\theta}_i \frac{N_i}{1 + N_i} > 0$ it follows that \bar{W} is a primitive matrix, i.e. it has only one eigenvalue of maximum modulus (see Theorem 8.5.2, p. 516 and Theorem 8.5.10, p. 520 in [31]).

It is obvious that $\ell^T = [1, \dots, 1]$ is the right eigenvector of \bar{W} corresponding to $\lambda_1 = 1$. Let y_W be the left eigenvector associated to the $\lambda_1 = 1$ eigenvalue and normalized so that $\ell^T y_W = 1$. Based on the above discussion matrix \bar{W} can be decomposed as follows:

$$\bar{W} = W_1 + \ell^T y_W \quad (62)$$

where

$$W_1 \ell = 0, \quad W_1^T y_W = 0, \quad \text{and} \quad \rho(W_1) < 1 \quad (63)$$

with $\rho(W_1)$ being the spectral radius of W_1 . We now show that the proposed algorithm provides strict-sense consensus.

Theorem 1. Let Assumptions A1–A3 hold. Then

$$(1) \lim_{t \rightarrow \infty} v_i(t) = v_c, \quad |v_c| < \infty, \quad 1 \leq i \leq N \quad (64)$$

$$(2) \lim_{t \rightarrow \infty} (x_j(t) - x_i(t)) = \bar{x}_{ij}, \quad |\bar{x}_{ij}| < \infty \quad (65)$$

for all $1 \leq i, j \leq N$.

Proof. Let

$$z(t+1) = v(t+1) - \ell y_W^T v(t) \quad (66)$$

where y_W is the same as in Eq. (62). We first show that

$$\|z(t+1)\| \leq c_{14} \rho_2^t, \quad \forall t \geq 0 \quad (67)$$

for some $0 < \rho_2 < 1$ and $0 < c_{14} < \infty$. Then we prove that statement (64) follows from (67). After substituting (62) in (58) we obtain

$$v(t+1) = W_1 v(t) + \ell y_W^T v(t) + \tilde{W}(t) v(t). \quad (68)$$

Since $W_1 \ell = 0$ and $\tilde{W}(t)\ell = (W(t) - \bar{W})\ell = 0$, Eqs. (66) and (68) imply

$$z(t+1) = W_1 z(t) + \tilde{W}(t)z(t). \quad (69)$$

Then

$$z(t+1) = T(t, 0)z(0) \quad (70)$$

where $T(t, 0) = \prod_{k=0}^t (W_1 + \tilde{W}(k))$. Using the fact that $\tilde{W}(t) \rightarrow 0$ as $t \rightarrow \infty$, and $\rho(W_1) = 1 - \epsilon_1$ for some $0 < \epsilon_1 < 1$, we can conclude that the transition matrix $T(t, 0)$ satisfies $\|T(t, 0)\| \leq c_{15}(1 - \epsilon_2)^t$ for some finite c_{15} and $0 < \epsilon_2 < 1$ (see for example Lemma A.2.13, p. 310 in [33]). Hence from (70) one obtains $\|z(t+1)\| \leq c_{15}(1 - \epsilon_2)^t$, $0 < \epsilon_2 < 1$ for all $t \geq 0$. We next prove statement (64). Note that from (66)

$$v(t+1) = P_1^{t+1}v(0) + \sum_{k=0}^t P_1^{t-k}z(k+1) \quad (71)$$

where $P_1 = \ell y_W^T$ and is an idempotent matrix, i.e. $P_1^k = P_1$, $k \geq 1$. Then from (71) we have

$$v(t+1) = P_1 v(0) + P_1 \sum_{k=0}^{t-1} z(k+1) + z(t+1). \quad (72)$$

Since by (67) $\sum_{k=0}^{\infty} z(k+1)$ is an absolutely convergent series, Eq. (72) implies that $\lim_{t \rightarrow \infty} v(t)$ exists. Let $\lim_{t \rightarrow \infty} v(t) = \bar{v}$, $\|\bar{v}\| < \infty$. Then from (30) and (37) we can derive $\lim_{t \rightarrow \infty} \ell^T \phi_i(t) = \lim_{t \rightarrow \infty} \ell^T (v(t) - v_i(t)\ell) = \ell^T \bar{v} - N \lim_{t \rightarrow \infty} v_i(t) = 0$ or $\lim_{t \rightarrow \infty} v_i(t) = \frac{1}{N} \ell^T \bar{v}$. This proves statement (64). Next we show the validity of (65). From Eq. (1) it follows that

$$x_i(t+1) - x_j(t+1) = x_i(0) - x_j(0) + \sum_{k=0}^t (v_i(k) - v_j(k)) \quad (73)$$

for all $i, j \in \mathcal{V}$. Obviously we need to demonstrate that the partial sum on the RHS of (73) is convergent. From (66) we can write $z_i(t+1) = v_i(t+1) - y_W^T v(t)$, $i \in \mathcal{V}$ where $z_i(t)$ is the i th component of the vector $z(t)$. Hence $v_i(t+1) - v_j(t+1) = z_i(t+1) - z_j(t+1)$ for all $i, j \in \mathcal{V}$. Then from (67) one obtains $|v_i(t+1) - v_j(t+1)| \leq c_{16} \rho_2^t$, $\forall t \geq 0$, and $0 < \rho_2 < 1$, which together with (73) gives (65). Thus the theorem is proved. \square

5. Simulation experiment

Consider a network of six agents characterized by a directed graph whose topology is defined by the following adjacency matrix

$$A_d = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

$A_d(i, j) = 1$ signifies that agent i directly receives information from agent j . $A_d(i, j) = 0$ means that agent i cannot receive any information from agent j . Let $\beta^T = [\beta_1, \dots, \beta_N]$ be a vector with β_i being parameters from Eq. (2). In our simulation experiment we take $\beta^T = [3.8, -1.5, 2.3, -0.7, 1.5, -3.4]$. Initial states of the model (1) and (2) are selected as $x_i(0) = 0.5(-2)^{i+1}$, and $v_i(0) = i(-1)^i$, $i = 1, \dots, 6$. In Eq. (18) the algorithm step size is set to $\mu_i = 0.4$, $i = 1, \dots, 6$. Fig. 1 depicts the convergence of the parameter estimates $\{\theta_i(t)\}$, $i = 1, \dots, 6$. Fig. 2 shows that all velocities $v_i(t)$, $i = 1, \dots, 6$ converge to the same value. Fig. 3 illustrates that the distance between the first and the fifth agent converges to a constant.

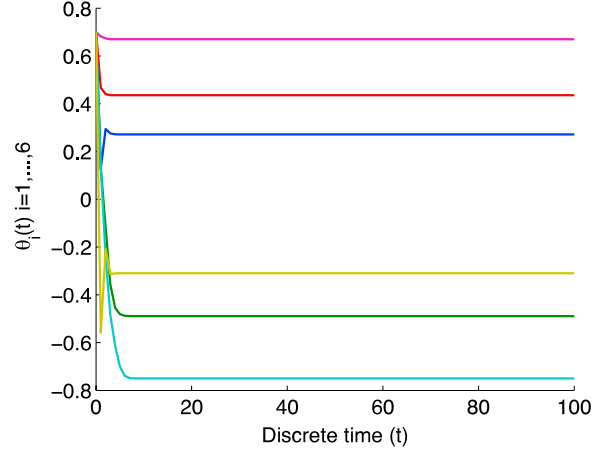


Fig. 1. Convergence of coupling parameters.

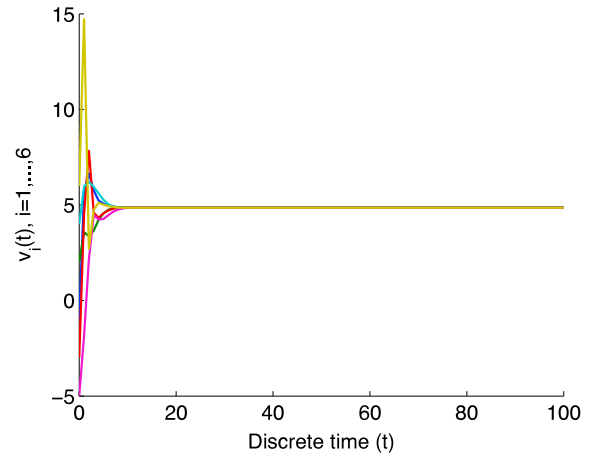


Fig. 2. Convergence of agent velocities $v_i(t)$.

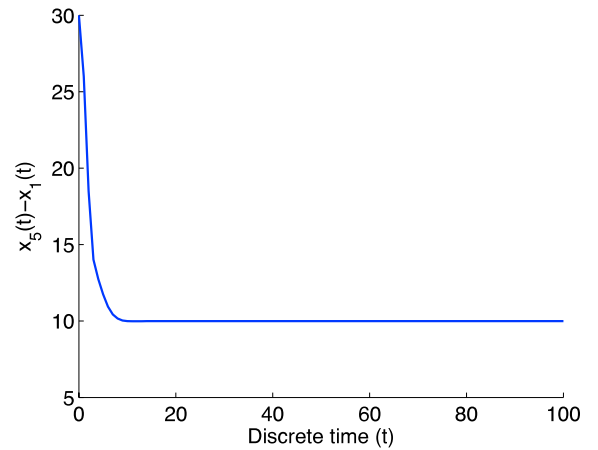


Fig. 3. Evolution of distance between the 1st and 5th agent.

6. Conclusion

Inspired by the evolution of flocks, we considered a multi-agent system where the coupling parameters are locally tuned so that agent velocities converge toward the same value. The tuning algorithm is based on a recursive normalized gradient scheme. Assuming that the graph is strongly connected, it is proved that the system achieves wide-sense consensus, and the sequence of the coupling parameters is convergent. Under additional constraints specified by Assumption A3, it is shown that all agent velocities

converge toward the same limit, thus achieving strict sense consensus. As a future research topic it is of interest to examine the behavior of the described NGA in the case of time-varying graph topology, presence of noise and delay in transmission channels between graph nodes, and robustness with respect to modeling errors of agent dynamics. As pointed out in [34,35] the kick model, also known as the model of pulse coupled oscillators, is prevalent in natural manifestations of rhythmic behavior in contrast to diffusive synchronization. Of particular interest is to investigate the possible application of the proposed self-tuning algorithm on the kick problem, as well as the problems of network clock synchronization [36,37].

References

- [1] A. Winfree, *Biological rhythms and the behavior of populations of coupled oscillators*, *J. Theoret. Biol.* 16 (1) (1967) 15–42.
- [2] Y. Kuramoto, Self-entrainment of a population of coupled non-linear oscillators, in: *Int. Symp. on Math. Problems in Theor. Phys.*, vol. 39, 1975, pp. 420–422.
- [3] S.H. Strogatz, From Kuramoto to Crawford: exploring the onset of synchronization in populations of coupled oscillators, *Physica D* 143 (1) (2000) 1–20.
- [4] J.A. Acebron, L. Bonilla, C. Vicente, F. Ritort, R. Spigler, The Kuramoto model: a simple paradigm for synchronization phenomena, *Rev. Modern Phys.* 77 (1) (2005) 137–185.
- [5] F. Dorfler, F. Bullo, Synchronization in complex oscillator networks: a survey, *Automatica* 50 (2014) 1539–1564.
- [6] R. Olfati-Saber, J.A. Fax, R.M. Murray, Consensus and cooperation in networked multi-agent systems, *Proc. IEEE* 95 (1) (2007) 215–233.
- [7] M.H. DeGroot, Reaching a consensus, *J. Amer. Statist. Assoc.* 69 (345) (1974) 118–121.
- [8] T. Vicsek, A. Czirok, E. Ben-Jacob, I. Cohen, O. Shochet, Novel type of phase transition in a system of self-driven particles, *Phys. Rev. Lett.* 75 (6) (1995) 1226–1229.
- [9] A. Jadbabaie, J. Lin, A.S. Morse, Coordination of groups of mobile autonomous agents using nearest neighbor rules, *IEEE Trans. Automat. Control* 48 (6) (2003) 988–1001.
- [10] R. Olfati-Saber, R.M. Murray, Consensus problems in networks of agents with switching topology and time-delays, *IEEE Trans. Automat. Control* 49 (9) (2004) 1520–1533.
- [11] F. Garin, L. Schenato, A survey on distributed estimation and control applications using linear consensus algorithms, in: *Networked Control Systems*, Springer-Verlag, London, UK, 2010, pp. 75–107.
- [12] E. Lovisari, S. Zampieri, Performance metrics in the average consensus problem: a tutorial, *Annu. Rev. Control* 36 (1) (2012) 26–41.
- [13] Y. Cao, W. Yu, W. Ren, G. Chen, An overview of recent progress in the study of distributed multi-agent coordination, *IEEE Trans. Ind. Inf.* 9 (1) (2013) 427–438.
- [14] W. Ren, Y. Cao, *Distributed Coordination of Multi-agent Networks: Emergent Problems, Models, and Issues*, Springer-Verlag, 2011.
- [15] Z. Qu, *Cooperative Control of Dynamical Systems: Applications to Autonomous Vehicles*, Springer-Verlag, 2009.
- [16] M. Mesbahi, M. Egerstedt, *Graph Theoretic Methods in Multiagent Networks*, Princeton University Press, 2010.
- [17] J. Zhou, J. Lu, J. Lü, Adaptive synchronization of an uncertain complex dynamical network, *IEEE Trans. Automat. Control* 51 (4) (2006) 652–656.
- [18] J. Zhou, J. Lu, J. Lü, Pinning adaptive synchronization of a general complex dynamical network, *Automatica* 44 (2008) 996–1003.
- [19] P. DeLellis, M. diBernardo, F. Garofalo, Novel decentralized adaptive strategies for synchronization of complex networks, *Automatica* 45 (2009) 1312–1318.
- [20] H. Bai, M. Arcak, J. Wen, Adaptive motion coordination: using relative velocity feedback to track a reference velocity, *Automatica* 45 (2009) 1020–1025.
- [21] A. Das, F. Lewis, Distributive adaptive control for synchronization of unknown nonlinear networked systems, *Automatica* 46 (2010) 2014–2021.
- [22] H. Yu, X. Xia, Adaptive consensus for multi-agents in networks with jointly connected topologies, *Automatica* 48 (2012) 1783–1790.
- [23] Z. Li, W. Ren, X. Liu, L. Xie, Distributed consensus of linear multi-agent systems with adaptive dynamic protocols, *Automatica* 49 (2013) 1986–1995.
- [24] Z. Li, W. Ren, X. Liu, M. Fu, Consensus of multi-agent systems with general linear and Lipschitz nonlinear dynamics using distributed adaptive protocols, *IEEE Trans. Automat. Control* 58 (7) (2013) 1786–1791.
- [25] J. Mei, W. Ren, J. Chen, Consensus of second order heterogeneous multi-agent system under a directed graph, in: *Proc. of the 2014 American Control Conference*, Portland, Oregon, 2014, pp. 802–807.
- [26] W. Chen, X. Li, W. Ren, C. Wen, Adaptive consensus of multi-agent systems with unknown identical control directions based on a novel Nussbaum-type function, *IEEE Trans. Automat. Control* 59 (7) (2014) 1887–1892.
- [27] H. Yu, Y. Shen, X. Xia, Adaptive finite time consensus in multi-agent networks, *Systems Control Lett.* 62 (2013) 880–889.
- [28] Y. Liu, H. Min, Z. Liu, S. Liao, Distributed adaptive consensus for multiple mechanical systems with switching topologies and time-varying delay, *Systems Control Lett.* 64 (2014) 119–126.
- [29] P. Junmin, Y. Xudong, Cooperative control of multiple heterogeneous agents with unknown high-frequency-gain signs, *Systems Control Lett.* 68 (2014) 51–56.
- [30] R.R. Gantmacher, *The Theory of Matrices*. Vol. 2, Chelsea Publishing Company, New York, 1959.
- [31] R.A. Horn, C.R. Johnson, *Matrix Analysis*, Cambridge University Press, 1996.
- [32] C.G. Goodwin, K. Sang Sin, *Adaptive Filtering, Prediction and Control*, Prentice Hall, Englewood Cliffs, NJ, USA, 1984.
- [33] J.W. Polderman, I. Mareels, *Adaptive Systems—An Introduction*, Birkhäuser, Boston, 1996.
- [34] A. Mauroy, P. Sacre, R. Sepulchre, Kick synchronization versus diffusive synchronization, in: *Proc. of the IEEE Conf. on Decision and Control*, Hawaii, December 2012, pp. 7171–7183.
- [35] A. Mauroy, R. Sepulchre, Global analysis of a continuum model for monotone pulse-coupled oscillators, *IEEE Trans. Automat. Control* 58 (5) (2013) 1154–1166.
- [36] G. Sayboth, F. Allgover, Clock synchronization over directed graphs, in: *Proc. of the 52nd IEEE Conf. on Decision and Control*, Florence, Italy, 2013, pp. 6105–6111.
- [37] R. Carli, S. Zampieri, Network clock synchronization based on the second-order linear consensus algorithm, *IEEE Trans. Automat. Control* 59 (2) (2014) 409–422.