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A chattering-free robust adaptive sliding mode controller for synchronization of two different chaotic systems with unknown uncertainties and external disturbances

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ABSTRACT

In this paper, a robust adaptive sliding mode controller (RASMC) is introduced to synchronize two different chaotic systems in the presence of unknown bounded uncertainties and external disturbances. The structure of the master and slave chaotic systems has no restrictive assumption. Appropriate adaptation laws are derived to tackle the uncertainties and external disturbances. Based on the adaptation laws and Lyapunov stability theory, an adaptive sliding control law is designed to ensure the occurrence of the sliding motion even when both master and slave systems are perturbed with unknown uncertainties and external disturbances. Since the conventional sliding mode controllers contain the sign function, the undesirable chattering is occurred. We propose a new simple adaptive scheme to eliminate the chattering. Finally, numerical simulations are presented to verify the usefulness and applicability of the proposed control strategy.

1. Introduction

Chaos is a very complex nonlinear phenomenon that exhibits some specific features such as crucially dependence to initial conditions, broad Fourier transform spectra, strange attractors and fractal properties of the motion in phase space. Because of these features, chaos synchronization has attracted an increasing interest among scientists of different fields and has found wide variety applications in physics and engineering including biological systems, chemical reactions, human heartbeat regulation, ecological systems, secure communication, information processing and so on [1]. Due to the extensive applications of chaos synchronization and since the pioneering work by Pecora and Carroll in 1990 [2], various control methods have been developed for synchronization of chaotic systems such as optimal control [3], sliding mode control [4], PID control [5], linear state feedback control [6], adaptive control [7], LMI-based non-fragile control [8], impulsive control [9], backstepping design [10], passive control [11], delayed feedback control [12–14], etc.

However, most of the mentioned works have focused on chaos synchronization between two chaotic systems without considering the effects of both uncertainties and external disturbances. While, in real-life practical applications there are usually unknown uncertainties and external disturbances in the systems' dynamics due to un-modeled dynamics, structural variations in plants, modeling errors and measurement and environment noises. Therefore, in the recent years, investigation of the problem of synchronizing two chaotic systems in the presence of uncertainties and external disturbances has become an interesting and important research topic. In this regard, some researchers have proposed different techniques for synchronizing uncertain

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chaotic systems which include fuzzy control [15,16], fuzzy sliding mode control [17], sliding mode control [18–20], H_∞ approach [21], active control [22], linear feedback control [23–25], nonlinear feedback control [26], backstepping method [27] and observer-based approaches [28–30].

Unfortunately, most of the previous techniques have been proposed for synchronizing two identical chaotic systems with different initial conditions. Whereas, in many practical situations, there is no two chaotic systems with complete identical structures. Moreover, increasingly applications of chaos synchronization in secure communications make it much more significant to synchronize two different chaotic systems [31]. Recently, some nonlinear control techniques have been used to synchronize two different chaotic systems without uncertainties and external disturbances [31–35]. However, there are few works about the problem of chaos synchronization between two different chaotic systems with uncertainties and external disturbances. Cai et al. [36] have reported modified projective chaos synchronization between two different chaotic systems with external disturbances. Yan et al. [37] have designed an adaptive sliding mode controller to synchronize a chaotic system (as the master system), which includes unknown parameters and external disturbances, and a deterministic linear system (as the slave system). Kebriaei and Yazdanpanah [38] have designed an adaptive sliding mode controller for synchronizing two different uncertain chaotic systems with input nonlinearities. The proposed controller is applicable only for the synchronization of chaotic systems in the canonical form with input nonlinearities. Yau [39] have derived an adaptive sliding mode controller for synchronization of two identical chaotic systems in canonic form and with known bounded uncertainties and disturbances. While, in real-life situations it is difficult to determine the bounds of the uncertainties and external disturbances in advance. In addition, they have supposed that only one of the state equations of the systems is perturbed by the uncertainties and external disturbances. However, in practice, the uncertainties and external disturbances affect the whole dynamics of the systems. In conclusion, investigation of the generic problem of chaos synchronization between two different chaotic systems with no restrictive assumption in the structure of the systems and with uncertainties and external disturbances, which is inherent in practical and real applications, has not been well discussed to this date.

The sliding mode control (SMC) [40] approach is a powerful and robust tool for controlling high-order nonlinear complex dynamical systems operating under various uncertainty conditions. The SMC has several useful advantages such as fast response, low sensitivity to external disturbances, robustness to the plant uncertainties and easy realization. In the SMC approach, once the system states reach to the sliding manifold, the system behavior is determined by the sliding surface dynamics. Therefore, the SMC decouples overall system motion into independent partial components of lower dimension, which decreases the complexity of the controller design.

In this paper, a robust adaptive sliding mode controller (RASMC) is designed to synchronize two different chaotic systems with parametric uncertainties and external disturbances. The structure of the master and slave systems is assumed to be quite general with no restrictive assumption. Unknown bounded uncertainties and external disturbances are added to the whole dynamics of both master and slave systems. Appropriate adaptation laws are derived to tackle the uncertainties and external disturbances. Since the conventional sliding mode controllers include the *sign* function, as a rigid switcher, they suffer from the undesirable chattering phenomenon. Some methods for chattering alleviation have been reviewed in [41]. However, in this paper, another new simple efficient technique for chattering avoidance is introduced:the discontinuous *sign* function in the control law is replaced by the continuous *tanh* function with an adaptive scheme to tune the amplitude and steepness of the function. Consequently, based on the adaptation laws and adaptive *tanh* function, a chattering-free RASMC is designed to guarantee the existence of the sliding motion even when the uncertainties and external disturbances are present in the systems' dynamics. The robustness and stability of the proposed RASMC are proved using Lyapunov stability theory. Some numerical simulations are presented to justify the efficiency and feasibility of the introduced RASMC.

2. System definition and problem formulation

In this paper, two n-dimensional master and slave chaotic systems with uncertainties and external disturbances are given as follows:

Master system:

$$\dot{x}(t) = f(x) + \Delta f(x,t) + d^m(t). \tag{1}$$

Slave system:

$$\dot{y}(t) = g(y) + \Delta g(y,t) + d^s(t) + u(t), \tag{2}$$

where $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \in R^{n\times 1}$, $\Delta f(x,t) = [\Delta f_1(x,t), \Delta f_2(x,t), \ldots, \Delta f_n(x,t)]^T \in R^{n\times 1}$ and $d^m(t) = \left[d_1^m(t), d_2^m(t), \ldots, d_n^m(t)\right]^T \in R^{n\times 1}$ are the vector of the states, uncertainties and external disturbances of the master system, respectively; $y(t) = [y_1(t), y_2(t), \ldots, y_n(t)]^T \in R^{n\times 1}$, $\Delta g(y,t) = [\Delta g_1(y,t), \Delta g_2(y,t), \ldots, \Delta g_n(y,t)]^T \in R^{n\times 1}$ and $d^s(t) = \left[d_1^s(t), d_2^s(t), \ldots, d_n^s(t)\right]^T \in R^{n\times 1}$ are the vectors of the states, uncertainties and external disturbances of the slave system, respectively; $f_i(x)$ and $g_i(y)$, $i = 1, 2, \ldots, n$ are continuous smooth nonlinear functions and $u(t) = [u_1(t), u_2(t), \ldots, u_n(t)]^T \in R^{n\times 1}$ is the vector of the control inputs to be designed later.

Assumption 1. The uncertainties $\Delta f(x,t)$ and $\Delta g(y,t)$ are assumed to be bounded. Therefore, there exist appropriate positive constants α_i^m, α_i^s and α_i, $i = 1,2,\ldots,n$ such that:

$$|\Delta f_i(x,t)| < \alpha_i^m \quad \text{and} \quad |\Delta g_i(x,t)| < \alpha_i^s, \quad i = 1,2,\ldots,n, \tag{3}$$

$$|\Delta f_i(x,t)| - \Delta g_i(x,t)| < \alpha_i, \quad i = 1,2,\ldots,n. \tag{4}$$

Assumption 2. It is assumed that the external disturbances are norm-bounded in C^1, i.e.

$$\left|d_i^m(t)\right| < \beta_i^m \quad \text{and} \quad \left|d_i^s(t)\right| < \beta_i^s, \quad i = 1,2,\ldots,n. \tag{5}$$

Consequently, one can obtain that:

$$\left|d_i^m(t)\right| < \beta_i^m \quad \text{and} \quad \left|d_i^s(t)\right| < \beta_i^s, \quad i = 1,2,\ldots,n. \tag{6}$$

Assumption 3. The constants $\alpha_i^m, \alpha_i^s, \alpha_i, \beta_i^m, \beta_i^s$ and β_i, $i = 1,2,\ldots,n$ are unknown in advance.

To solve the synchronization problem, the error between the master and slave systems is defined as $e(t) = x(t) - y(t)$. Subtracting Eq. (2) from Eq. (1), we obtain the synchronization error dynamics as follows:

$$\begin{aligned}
\dot{e}_1(t) &= f_1(x) + \Delta f_1(x,t) + d_1^m(t) - g_1(y) - \Delta g_1(y,t) - d_1^s(t) - u_1(t),\\
\dot{e}_2(t) &= f_2(x) + \Delta f_2(x,t) + d_2^m(t) - g_2(y) - \Delta g_2(y,t) - d_2^s(t) - u_2(t),\\
&\vdots\\
\dot{e}_n(t) &= f_n(x) + \Delta f_n(x,t) + d_n^m(t) - g_n(y) - \Delta g_n(y,t) - d_n^s(t) - u_n(t).
\end{aligned} \tag{7}$$

It is clear that the synchronization problem is transformed to the equivalent problem of stabilization of the error system (7). The main objective of this paper is to design a feedback control law for any given master chaotic system (1) and slave chaotic system (2) with uncertainties and external disturbances such that the asymptotical stability of the resulting error system (7) can be achieved in the sense that $\lim_{t\to\infty}\|e(t)\| = 0$ or equivalently $x(t) \to y(t)$ as $t \to \infty$.

3. Design of robust adaptive sliding mode controller

The design procedure of a sliding mode controller has two stages. The first stage is to select a switching surface with a desired behavior. Therefore, a suitable sliding surface for application is defined as:

$$s_i(t) = \lambda_i e_i(t), \quad i = 1,2,\ldots,n, \tag{8}$$

where $s_i(t) \in R$, $i = 1,2,\ldots,n$ ($S(t) = [s_1(t), s_2(t), \ldots, s_n(t)]^T$) and the sliding surface parameters λ_is are selected to get positive values.

The second stage of the sliding mode controller design procedure is to determine a sliding control law to force the system trajectories onto the sliding surface and to maintain the system trajectories on it for the subsequent time. Basically, the sliding control law includes a continuous control law (called equivalent control) which controls the system trajectories on the sliding surface and a discontinuous control law (including the *sign* function) which handles the uncertainties and causes the chattering phenomenon. Therefore, to guarantee the existence of the sliding motion (i.e. to satisfy the reaching condition $S^T(t)\dot{S}(t) \leqslant 0$) and to eliminate the chattering phenomenon caused by the discontinuous *sign* function, an adaptive continuous control law is proposed as:

$$u_i(t) = f_i(x) - g_i(y) + (\hat{\alpha}_i + \hat{\beta}_i + \mu_i)tanh(\psi_i s_i), \quad i = 1,2,\ldots,n, \tag{9}$$

where $\hat{\alpha} > 0$ and $\hat{\beta} > 0$ are two adaptive parameters to undertake the unknown uncertainty bounds $\hat{\alpha}_i$ and $\hat{\beta}_i$, respectively; $\mu_i > 0$ and $\psi_i > 0$ are adaptation coefficients which tune the gain and steepness of the *tanh* function, respectively.

Let suitable adaptation laws to be defined as follows:

$$\begin{aligned}
\dot{\hat{\alpha}}_i &= -\lambda_i|s_i|, \hat{\alpha}_i(0) = \hat{\alpha}_{i0} > 0,\\
\dot{\hat{\beta}}_i &= -\lambda_i|s_i|, \hat{\beta}_i(0) = \hat{\beta}_{i0} > 0,\\
\dot{\mu}_i &= -p_i|s_i||e_i|, \hat{\mu}_i(0) = \mu_{i0} > 0,\\
\dot{\psi}_i &= -q_i|s_i||e_i|, \hat{\psi}_i(0) = \psi_{i0} > 0,
\end{aligned} \tag{10}$$

where p_i and q_i are two positive constants and $\hat{\alpha}_{i0}, \hat{\beta}_{i0}, \mu_{i0}$ and ψ_{i0} are the initial values of the adaptation parameters $\hat{\alpha}_i, \hat{\beta}_i, \mu_i$ and ψ_i respectively.

Based on the control law in Eq. (9) and the adaptation laws in Eq. (10), to ensure the occurrence of the sliding motion, a theorem is proposed and proved. Before proceeding to the theorem, an auxiliary lemma is presented.

Lemma 1. *For every given scalar x and positive scalar y the following inequality holds:*

$$x\tan h(yx) = |x\tan h(yx)| = |x||\tan h(yx)| \geqslant 0. \tag{11}$$

Proof. From the mathematical definition of the *tanh*(.) function, we have

$$x\tan h(yx) = x\frac{e^{yx}-e^{-yx}}{e^{yx}+e^{-yx}}. \tag{12}$$

Multiplying the above equation by $\frac{e^{yx}}{e^{yx'}}$, one has

$$x\tan h(yx) = \left(\frac{1}{e^{2yx}+1}\right)x(e^{2yx}-1). \tag{13}$$

According to $\begin{cases}(e^{2yx}-1)\geqslant 0 & \text{if } x\geqslant 0\\ (e^{2yx}-1)<0 & \text{if } x<0\end{cases}$, one can obtain

$$x(e^{2yx}-1) \leqslant 0. \tag{14}$$

Based on $\left(\frac{1}{e^{2yx}+1}\right) > 0$ and from Eq. (14), we have

$$x\tan h(yx) = \left(\frac{1}{e^{2yx}+1}\right)x(e^{2yx}-1) \geqslant 0. \tag{15}$$

Therefore, from the fact that for every scalars z and v, if $zv \geqslant 0$ then $zv = |zv| = |z||v| \geqslant 0$ holds, one can conclude that

$$x\tan h(yx) = |x\tan h(yx)| = |x||\tan h(yx)| \geqslant 0. \tag{16}$$

This completes the proof. □

Theorem 1. *Consider the synchronization error system (7). If, this system is controlled by the continuous control law u(t) in Eq. (9) with the adaptation laws in Eq. (10), then the system trajectories will converge to the sliding surface S(t) = 0.*

Proof. Consider a positive definite Lyapunov function candidate in the following form:

$$V(t) = \frac{1}{2}\sum_{i=1}^{n}\left[s_i^2 + (\hat{\alpha}_i+\alpha_i)^2 + (\hat{\beta}_i+\beta_i)^2 + \mu_i^2 + \psi_i^2\right]. \tag{17}$$

Taking derivative of the Lyapunov function candidate with respect to time, one has

$$\dot{V}(t) = \sum_{i=1}^{n}[s_i\dot{s}_i + (\hat{\alpha}_i+\alpha_i)\dot{\hat{\alpha}}_i + (\hat{\beta}_i+\beta_i)\dot{\hat{\beta}}_i + \mu_i\dot{\mu}_i + \psi_i\dot{\psi}_i]. \tag{18}$$

Using $\dot{s}_i = \lambda_i\dot{e}_i$ and replacing $\dot{e}_i$ from (7) into the above equation, we have

$$\dot{V}(t) = \sum_{i=1}^{n}[s_i\lambda_i(f_i(x)+\Delta f_i(x,t)+d_i^m(t)-g_i(y)-\Delta g_i(y,t)-d_i^s(t)-u_i(t)) + (\hat{\alpha}_i+\alpha_i)\dot{\hat{\alpha}}_i + (\hat{\beta}_i+\beta_i)\dot{\hat{\beta}}_i + \mu_i\dot{\mu}_i + \psi_i\dot{\psi}_i]. \tag{19}$$

Introducing the control law (9) and the adaptation laws (10) into the right hand of (19), one obtains

$$\begin{aligned}\dot{V}(t) = \sum_{i=1}^{n}\Big[&s_i\lambda_i(f_i(x)+\Delta f_i(x,t)+d_i^m(t)-g_i(y)-\Delta g_i(y,t)-d_i^s(t)-(f_i(x)-g_i(y)+(\hat{\alpha}_i+\hat{\beta}_i+\mu_i)\tan h(\psi_i s_i)))\\ &-(\hat{\alpha}_i+\alpha_i)\lambda_i|s_i| - (\hat{\beta}_i+\beta_i)\lambda_i|s_i| - \mu_i p_i|s_i||e_i| - \psi_i q_i|s_i||e_i|\Big].\end{aligned} \tag{20}$$

It is clear that

$$\dot{V}(t) \leqslant \sum_{i=1}^{n}\Big[|s_i|\lambda_i(|\Delta f_i(x,t)-\Delta g_i(y,t)| + |d_i^m(t)-d_i^s(t)|) - \lambda_i s_i(\hat{\alpha}_i+\hat{\beta}_i+\mu_i)\tan h(\psi_i s_i) - (\hat{\alpha}_i+\alpha_i)\lambda_i|s_i| - (\hat{\beta}_i+\beta_i)\lambda_i|s_i| - \mu_i p_i|s_i||e_i| - \psi_i q_i|s_i||e_i|\Big]. \tag{21}$$

Using Assumptions 1 and 2, we have

$$\dot{V}(t) \leqslant \sum_{i=1}^{n}[\lambda_i|s_i|(\alpha_i+\beta_i) - \lambda_i s_i(\hat{\alpha}_i+\hat{\beta}_i+\mu_i)\tan h(\psi_i s_i) - (\hat{\alpha}_i+\alpha_i)\lambda_i|s_i| - (\hat{\beta}_i+\beta_i)\lambda_i|s_i| - \mu_i p_i|s_i||e_i| - \psi_i q_i|s_i||e_i|]. \tag{22}$$

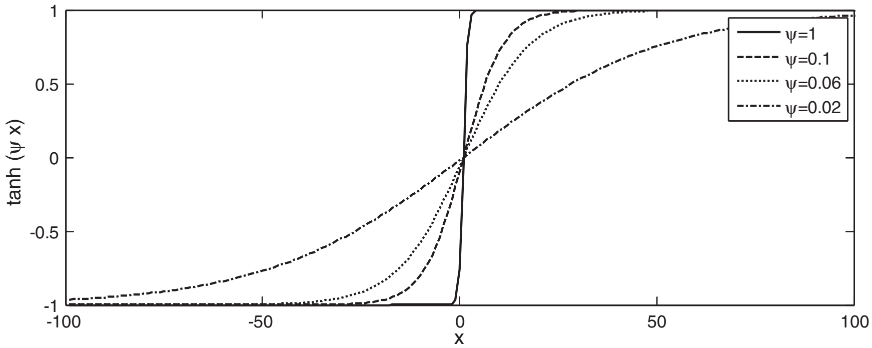

Fig. 1. Time response of the *tanh* function with several different steepness.

It is obvious that

$$\dot{V}(t) \leqslant \sum_{i=1}^{n}[-\lambda_i s_i(\hat{\alpha}_i + \hat{\beta}_i + \mu_i)\tan h(\psi_i s_i) - \hat{\alpha}_i\lambda_i|s_i| - \hat{\beta}_i\lambda_i|s_i| - \mu_i p_i|s_i||e_i| - \psi_i q_i|s_i||e_i|]. \quad (23)$$

From Lemma 1 and the fact that $\hat{\alpha}_i, \hat{\beta}_i, \lambda_i, \mu_i, \psi_i, p_i$ and q_i are all positive, one has

$$\dot{V}(t) \leqslant -\sum_{i=1}^{n}\left[|s_i|\underbrace{(\lambda_i(\hat{\alpha}_i + \hat{\beta}_i)(|\tan h(\psi_i s_i)| + 1) + \lambda_i\mu_i|\tan h(\psi_i s_i)| + \mu_i p_i|e_i| + \psi_i q_i|e_i|)}_{\eta_i}\right] = -\sum_{i=1}^{n}\eta_i|s_i| = -\mathrm{H}|S| \leqslant 0, \quad (24)$$

where $H = [\eta_1, \eta_2, \ldots, \eta_n] > 0$ and $|S| = [|s_1|, |s_2|, \ldots, |s_n|]^T$.

Therefore $\dot{V}(t)$ becomes

$$\dot{V}(t) = -\mathrm{H}|S| = -\omega(t) \leqslant 0, \quad (25)$$

where $\omega(t) = \mathrm{H}|S| \geqslant 0$. Integrating Eq. (25) from zero to t yields

$$V(0) \geqslant V(t) + \int_0^t \omega(\lambda)d\lambda. \quad (26)$$

Since $\dot{V}(t) \leqslant 0$, $V(0) - V(t) \geqslant 0$ is positive and finite, hence $\lim_{t\to\infty}\omega(\lambda)$ exists and is finite (i.e. $\lim_{t\to\infty}\omega(\lambda) = V(0) - V(t) \geqslant 0$). Thus, according to the Barbalat lemma [42], it can be concluded that

$$\lim_{t\to\infty}\omega(t) = \lim_{t\to\infty}H|S| = 0. \quad (27)$$

Since H is positive, Eq. (27) implies $S(t) = 0$. Thus the proof is achieved completely. □

Remark 1. Theorem 1 is also applicable for the chaos synchronization between two identical chaotic systems with unknown bounded uncertainties and external disturbances if the systems (1) and (2) satisfy $f(\cdot) = g(\cdot)$.

Remark 2. As mentioned before, the discontinuity of the *sign* function in the control law causes the chattering. Therefore, in order to avoid the chattering, the discontinuous *sign* function is replaced by the continuous *tanh* function with the adaptive gain and steepness. In other word, the function is used as an approximator of the *sign* function. As it can be seen in Fig. 1, the steepness of the *tanh* function determines that how the *tanh* can approximate the *sign* function. A larger steepness, a closer approximation to the *sign* function is obtained (i.e. the *tanh* function with a large steepness acts as the *sign* function and the chattering is taken place). On the other hand, it is well-known that the magnitude of the chattering is proportional to the *sign* function gain [41]. Thus, the chattering elimination idea is to reduce the steepness and gain of the continuous *tanh* function to remove the chattering preserving the existence of the sliding mode as proved in Theorem 1. To support this idea, an adaptively tuned function is used instead of the discontinuous *sign* function.

4. Numerical simulations

In this section, numerical simulations are presented to validate the robustness and effectiveness of the proposed RASMC. The ode45 solver of the MATLAB software is applied for solving differential equations. The Lorenz, Chen, and Genesio systems are three well-known chaotic systems with the following mathematical expressions.

$$\text{Lorenz}:\begin{cases}\dot{x}_1 = 10(x_2 - x_1),\\ \dot{x}_2 = 28x_1 - x_2 - x_1x_3,\\ \dot{x}_3 = x_1x_2 - 8/3x_3,\end{cases}$$

$$\text{Chen}:\begin{cases}\dot{y}_1 = 35(y_2 - y_1),\\ \dot{y}_2 = 28y_2 - 7y_1 - y_1y_3,\\ \dot{y}_3 = y_1y_2 - 3y_3,\end{cases}$$

$$\text{Genesio}:\begin{cases}\dot{z}_1 = z_2,\\ \dot{z}_2 = z_3,\\ \dot{z}_3 = -6z_1 - 2.92z_2 - 1.2z_3 + z_1^2.\end{cases} \tag{28}$$

Here, two different pairs of chaotic systems (Lorenz–Chen and Chen–Genesio) are synchronized using the proposed RASMC. In both cases, 0.6 cos t and $-$0.6 cos t, as the external disturbances, are added to the equations of the master and slave systems, respectively. Also, the following uncertainties are considered in the simulations.

$$\begin{cases}\Delta f_1(x) = 0.5\sin(\pi x_1),\\ \Delta f_2(x) = 0.5\sin(2\pi x_2),\\ \Delta f_3(x) = 0.5\sin(3\pi x_3)\end{cases} \quad \text{and} \quad \begin{cases}\Delta g_1(y) = -0.5\sin(\pi y_1),\\ \Delta g_2(y) = -0.5\sin(2\pi y_2),\\ \Delta g_3(y) = -0.5\sin(3\pi y_3).\end{cases} \tag{29}$$

For simplicity, it is assumed that $\mu_i = 0.001\psi_i$ and $p_i = q_i$, $i = 1, 2, 3$, and the initial values of μ_1, μ_2, and μ_3 are all set to 10. Three sliding surfaces are defined as $s_1 = 2e_1$, $s_2 = 2e_2$, and $s_3 = 2e_3$.

4.1. Robust chaos synchronization between the Lorenz and Chen systems

To show the efficiency of the proposed RASMC in synchronizing the Lorenz and Chen systems with unknown uncertainties and external disturbances it is assumed that the Lorenz system drives the Chen system. Therefore, the error dynamics using Eq. (7) can be obtained as:

$$\begin{cases}\dot{e}_1 = 35(e_2 - e_1) + 25(x_2 - x_1) + 0.5\sin(\pi x_1) + 0.5\sin(\pi y_1) + 1.2\cos t - u_1(t),\\ \dot{e}_2 = -7e_1 + 28e_2 - 35x_1 + 29x_2 + x_1x_3 - y_1y_3 + 0.5\sin(2\pi x_2) + 0.5\sin(2\pi y_2) + 1.2\cos t - u_2(t),\\ \dot{e}_3 = -3e_3 - 1/3x_3 - x_1x_2 + y_1y_2 + 0.5\sin(3\pi x_3) + 0.5\sin(3\pi y_3) + 1.2\cos t - u_3(t).\end{cases} \tag{30}$$

According to Theorem 1 and Eq. (9), the control inputs are derived as:

$$\begin{cases}u_1(t) = 35(e_2 - e_1) + 25(x_2 - x_1) + (\hat{\alpha}_1 + \hat{\beta}_1 + \mu_1)\tanh(2\psi_1 e_1),\\ u_2(t) = -7e_1 + 28e_2 - 35x_1 + 29x_2 + x_1x_3 - y_1y_3 + (\hat{\alpha}_2 + \hat{\beta}_2 + \mu_2)\tanh(2\psi_2 e_2),\\ u_3(t) = -3e_3 - 1/3x_3 - x_1x_2 + y_1y_2 + (\hat{\alpha}_3 + \hat{\beta}_3 + \mu_3)\tanh(2\psi_3 e_3).\end{cases} \tag{31}$$

Vectors [10,10,10] and [2,2,2] are selected as the initial conditions of the Lorenz and Chen systems, respectively. Furthermore, vectors [20,20,20] and [15,15,15] are selected as the initial values of the adaptation vector parameters $\hat{\alpha}$ and $\hat{\beta}$, respectively.

Fig. 2 shows the synchronization errors of the Lorenz and Chen systems, where the control inputs are applied at t = 5 s. As one can see, the synchronization errors converge to zero rapidly, which implies the chaos synchronization between the

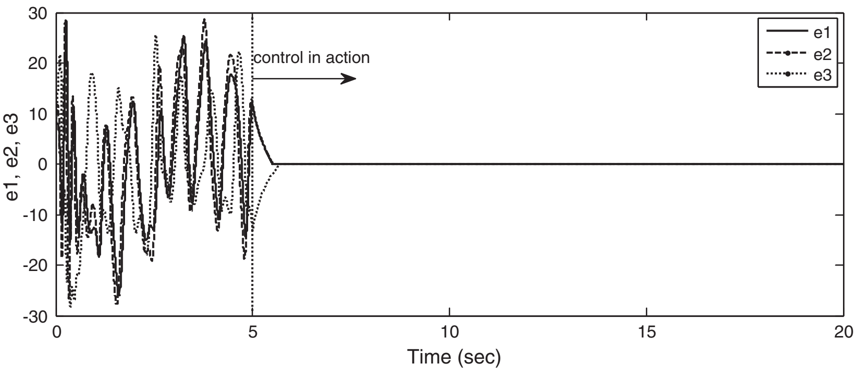

Fig. 2. Synchronization errors of the Lorenz and Chen systems.

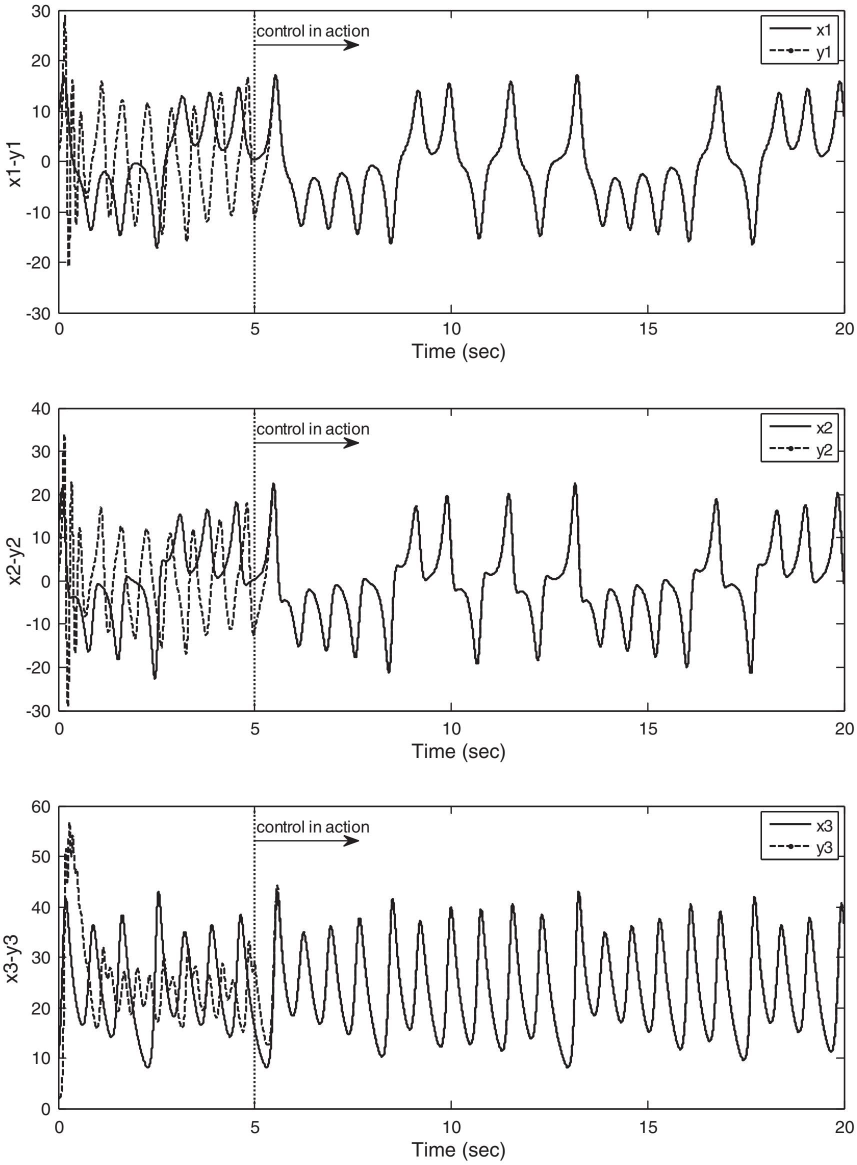

Fig. 3. State trajectories of the Chen and Genesio systems.

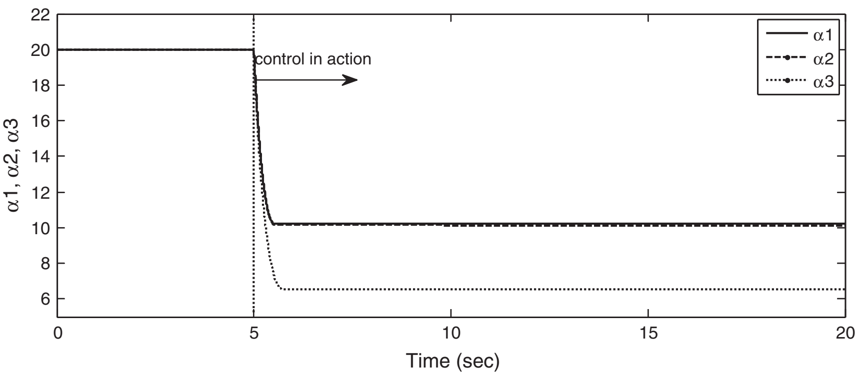

Fig. 4. Time response of the adaptation vector parameter $\hat{\alpha}$.

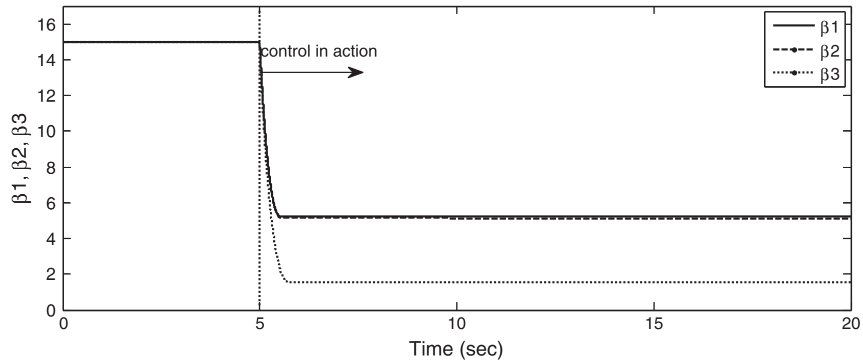

Fig. 5. Time response of the adaptation vector parameter $\hat{\beta}$.

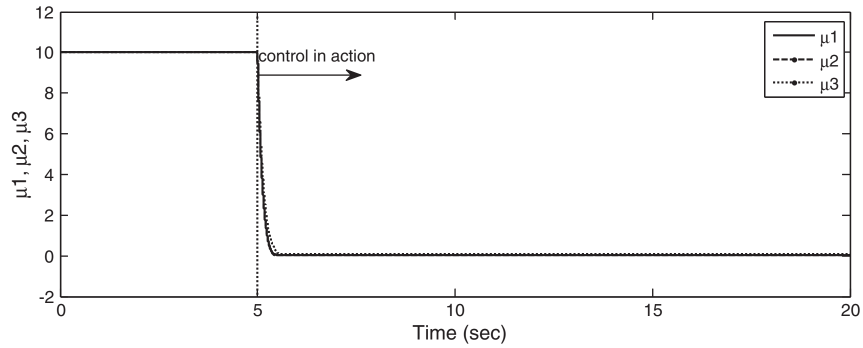

Fig. 6. Time response of the adaptation vector parameter μ.

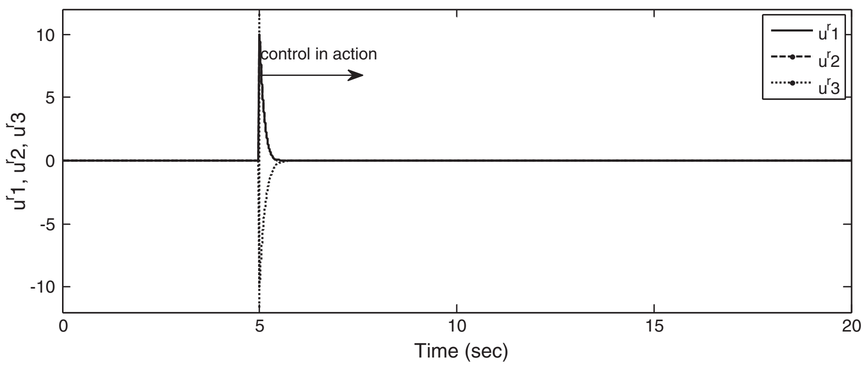

Fig. 7. Time response of the control input $u^r(t)$.

Lorenz and Chen systems is realized (as revealed in Fig. 3). The time responses of the adaptation vector parameters $\hat{\alpha}, \hat{\beta}$ and μ are depicted in Figs. 4–6, respectively. Obviously, all of the adaptation parameters converge to some constants. The continuous adaptive part of the control input $u_i^r(t) = \mu_i tanh(\psi_i s_i),\ i = 1, 2, 3$ is illustrated in Fig. 7. It is clear that the chattering phenomenon is completely removed.

4.2. Robust chaos synchronization between the Chen and Genesio systems

In this example, the Chen and Genesio chaotic systems are synchronized using the introduced RASMC. It is assumed that the Chen system is the master system and the Genesio system is the slave system. Therefore, the error dynamics using Eq. (7) is defined as:

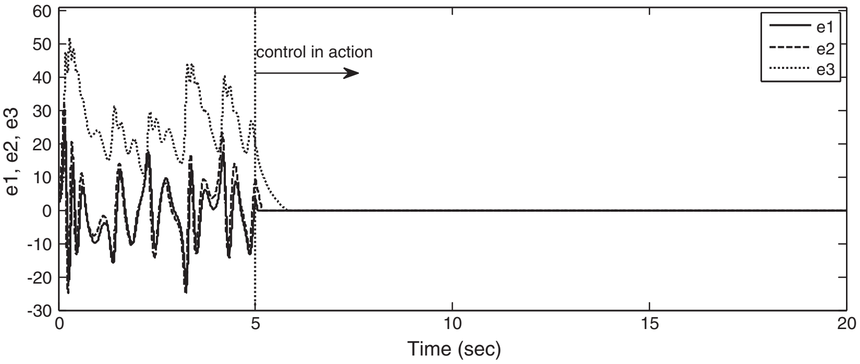

Fig. 8. Synchronization errors of the Chen and Genesio systems.

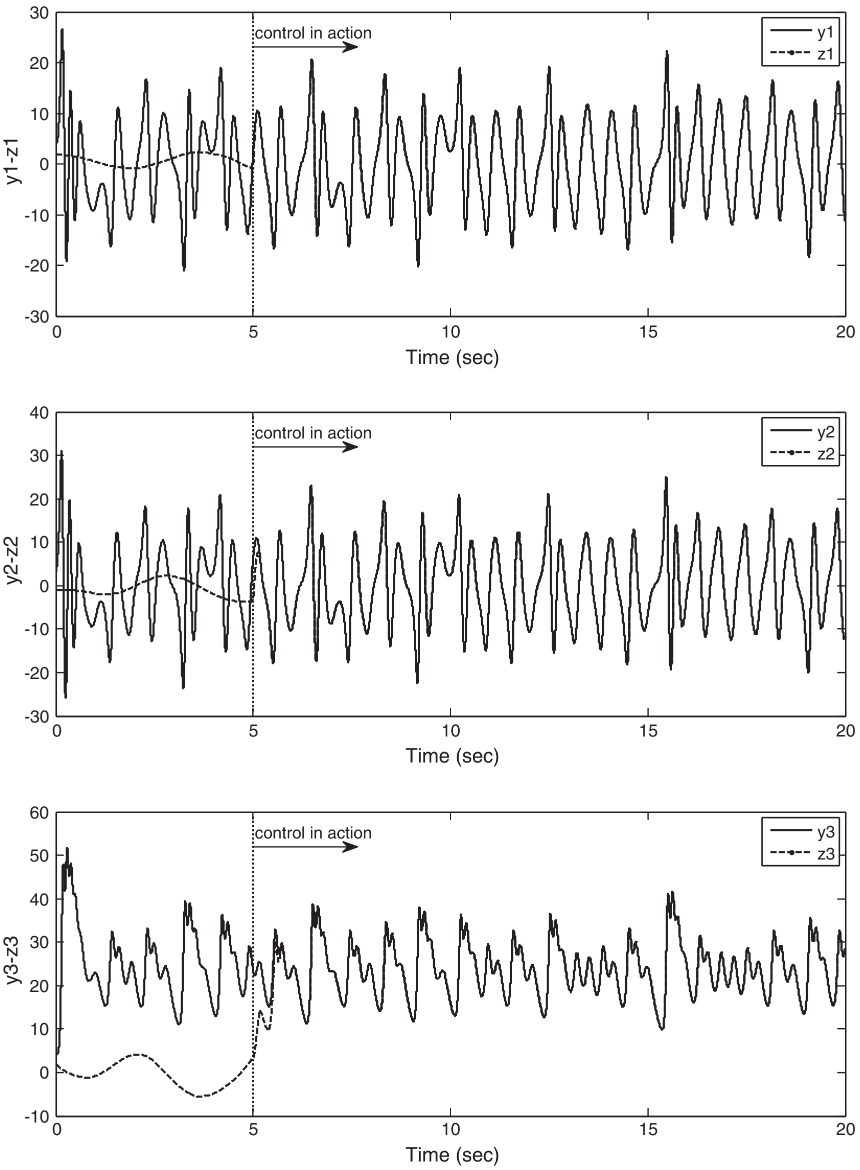

Fig. 9. State trajectories of the Chen and Genesio systems.

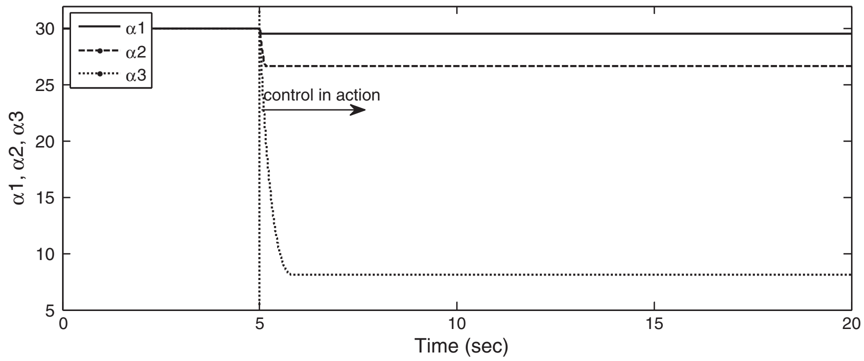

Fig. 10. Time response of the adaptation vector parameter $\hat{\alpha}$.

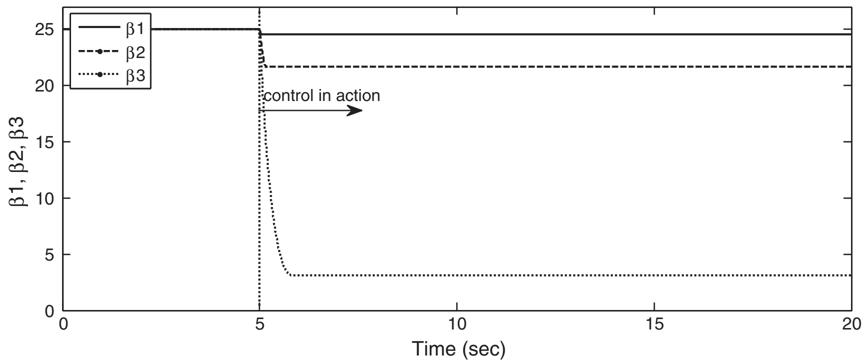

Fig. 11. Time response of the adaptation vector parameter $\hat{\beta}$.

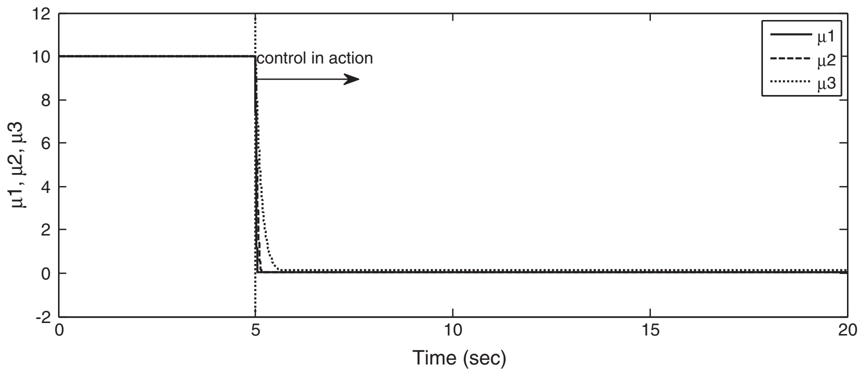

Fig. 12. Time response of the adaptation vector parameter μ.

$$\begin{cases} \dot{e}_1 = 35e_2 - 35y_1 + 34z_2 + 0.5\sin(\pi y_1) + 0.5\sin(\pi z_1) + 1.2\cos t - u_1(t), \\ \dot{e}_2 = -7y_1 + 28y_2 - y_1 y_3 - z_3 + 0.5\sin(2\pi y_2) + 0.5\sin(2\pi z_2) + 1.2\cos t - u_2(t), \\ \dot{e}_3 = -3e_3 + y_1 y_2 + 6z_1 + 2.92z_2 - 1.8z_3 - z_1^2 + 0.5\sin(3\pi y_3) + 0.5\sin(3\pi z_3) + 1.2\cos t - u_3(t). \end{cases} \quad (32)$$

Consequently, according to Theorem 1 and Eq. (9), the control inputs are developed as:

$$\begin{cases} u_1(t) = 35e_2 - 35y_1 + 34z_2 + (\hat{\alpha}_1 + \hat{\beta}_1 + \mu_1)\tan h(2\psi_1 e_1), \\ u_2(t) = -7y_1 + 28y_2 - y_1 y_3 - z_3 + (\hat{\alpha}_2 + \hat{\beta}_2 + \mu_2)\tan h(2\psi_2 e_2), \\ u_3(t) = -3e_3 + y_1 y_2 + 6z_1 + 2.92z_2 - 1.8z_3 - z_1^2 + (\hat{\alpha}_2 + \hat{\beta}_2 + \mu_2)\tan h(2\psi_3 e_3). \end{cases} \quad (33)$$

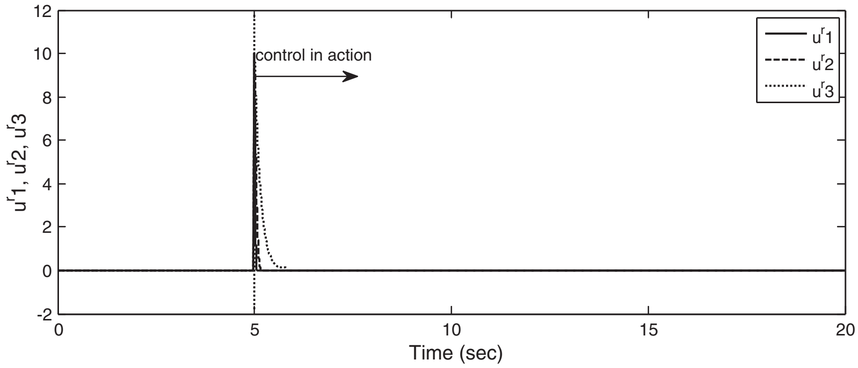

Fig. 13. Time response of the control input $u^r(t)$.

Vectors [5,3,4] and [2,−1,2] are selected as the initial conditions of the Chen and Genesio systems, respectively. Moreover, vectors [30,30,30] and [25,25,25] are chosen as the initial values of the adaptation vector parameters $\hat{\alpha}$ and $\hat{\beta}$, respectively.

The synchronization errors between the Chen and Genesio systems are illustrated in Fig. 8, while the control inputs are activated at $t = 5$. It can be seen that the synchronization errors converge to zero quickly, which indicates that the Chen and Genesio systems are indeed synchronized (as shown in Fig. 9). The time responses of the adaptation vector parameters $\hat{\alpha}, \hat{\beta}$ and μ are shown in Figs. 10–12, respectively. It is obvious that all adaptation parameters approach to some fixed values. The continuous adaptive part of the control input $u^r(t)$ is displayed in Fig. 13. One can see that the control $u^r(t)$ has no oscillations and attains zero.

Remark 3. It should be noted that the adaptive parameters $\hat{\alpha}_i$ and $\hat{\beta}_i$ are introduced to tackle the bounds of the error system's uncertainties and external disturbances α_i and β_i, respectively. On the other hand, according to the Lyapunov function in Eq. (17) and Theorem 1, one can conclude that the Lyapunov function will decrease as time goes to infinity and, therefore, $\hat{\alpha}_i$ and $\hat{\beta}_i$ will converge to $-\alpha_i$ and $-\beta_i$, respectively. This means that the adaptive parameters $\hat{\alpha}_i$ and $\hat{\beta}_i$ are the estimators of $-\alpha_i$ and $-\beta_i$, respectively. However, since the parameters $\hat{\alpha}_i$ and $\hat{\beta}_i$ are assumed to get positive values and are adaptively updated using the adaptation laws in Eq. (10), therefore, they converge to some positive constants (not to $-\alpha_i$ and $-\beta_i$), as illustrated in simulation results.

5. Conclusions

The problem of chaos synchronization between two different chaotic systems with unknown bounded uncertainties and external disturbances was investigated and solved using a novel robust adaptive chattering-free sliding mode controller. The structure of the master and slave systems was without any restrictive assumption. Both master and slave systems were perturbed by the unknown uncertainties and external disturbances. Suitable adaptation laws were derived to undertake the uncertainties and external disturbances. On the basis of Lyapunov stability theory and the adaptation laws, the proposed controller was designed. Undesirable chattering phenomenon was successfully alleviated using a simple adaptive scheme. The simulation results showed that the proposed controller works well for synchronizing two different chaotic systems even with the unknown uncertainties and external disturbances in both master and slave systems.

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